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Numerical simulations of neutron star crusts

Gundlach, Hawke & Erickson, *Classical Quantum Grav.* **29** (2012) 015005
Penner, Andersson, Jones, Samuelsson & Hawke, *ApJ* **749** (2012) L36
Millmore &, Hawke, *Classical Quantum Grav.* **27** (2010) 015007

I. Hawke, C. Gundlach, S. Erickson, J. Muddle, J. Penner, N. Andersson, D.I. Jones, L. Samuelsson, S. Millmore

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University of Southampton, UK

Cardiff, 30 November 2012

Outline

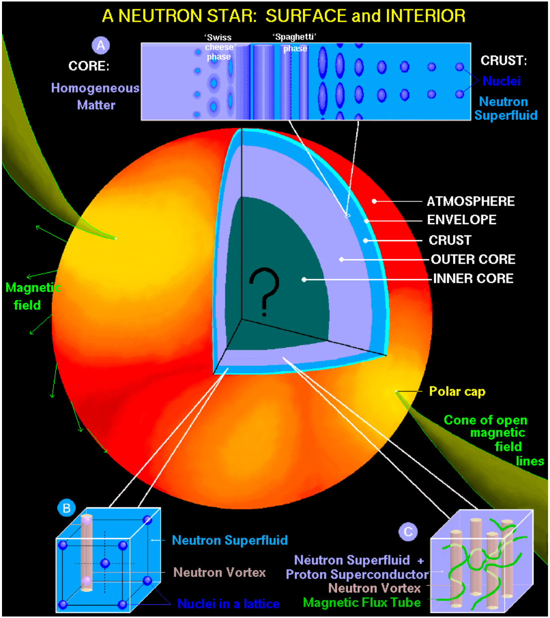
1. Background
   - Neutron stars
   - Context
   - Neutron star crust
2. Crustal evolution
   - Crustal evolution
3. Relasticity
   - Matter space
   - Equations of motion
   - Numerics
   - Numerical results
4. Interfaces
   - Interfaces
   - Level sets
   - Numerical results
5. Conclusions

Neutron stars

Neutron stars are laboratories for extreme physics.

More massive than the sun, diameter less than a large city.

Observations through electromagnetic and gravitational waves constrain matter and gravity theories: general relativity essential.



Context – numerical relativity

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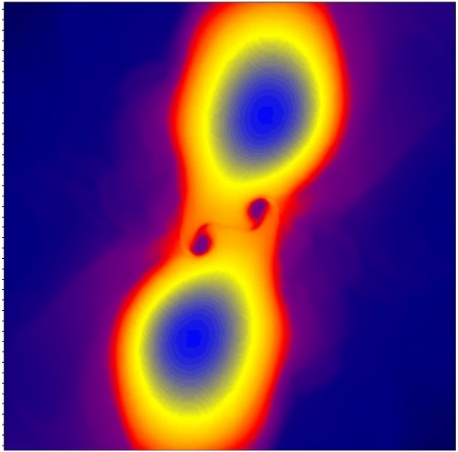
NS binary merger a major GW source for ground-based detectors.

Full nonlinear simulations needed for detailed template from late inspiral to early post-merger.

Current simulations

- expensive
- moderate accuracy
- missing input physics that may affect especially post-merger HMNS.



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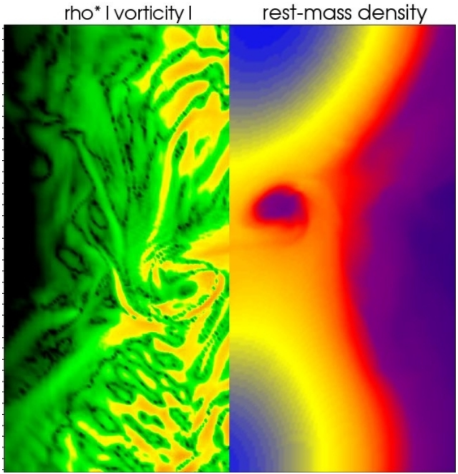
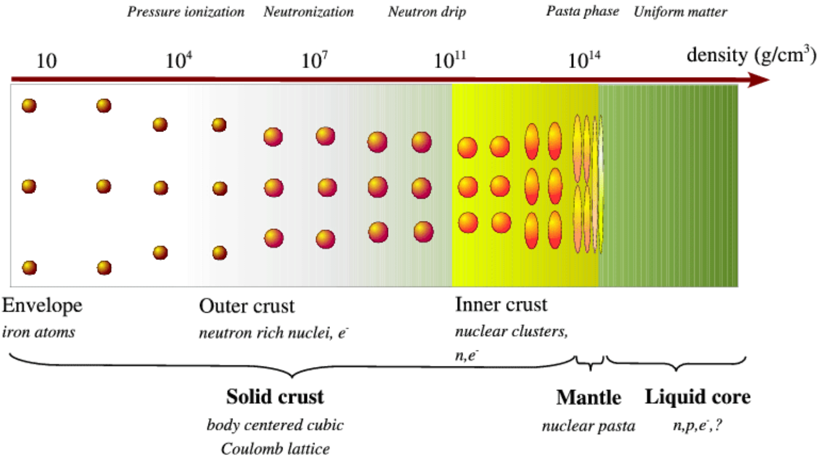


Crust structure



The bulk of a NS can be modelled as a (cold, magnetized) fluid.

But outer layers form a crystal lattice: the *crust*.

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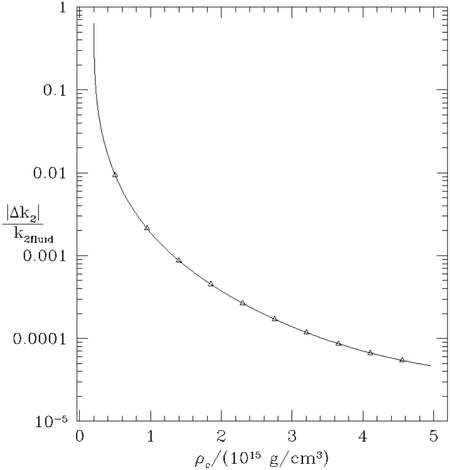
Evolution and crusts

Original belief: crusts negligible during inspiral, melt/crumble before merger.

Recent results (Penner et al.) show small corrections to Love number in inspiral. However, parts of crust survive to merger.

Suggested that crust may shatter via resonance (Tsang et al.).

Additional question: crust effect post-merger? (Figure from hydrodynamics simulation of Baiotti et al.)



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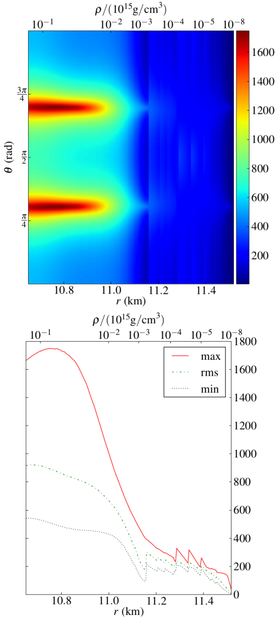

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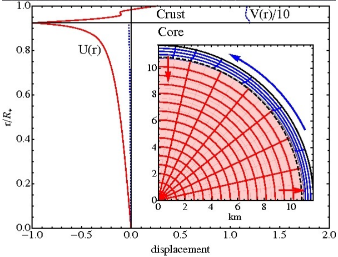

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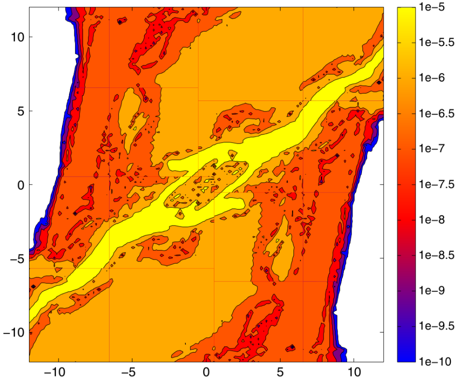

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Matter space

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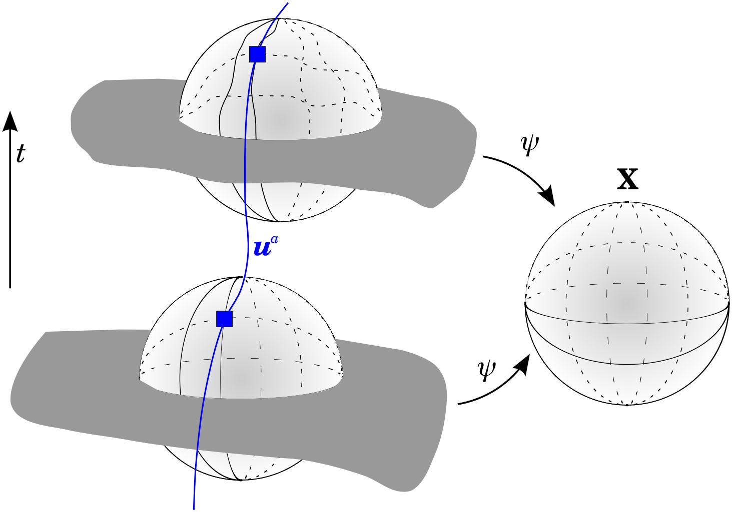

A *body* is given by a *reference configuration* X, and its deformation computed from the map ψ.

Equations of motion

Matter space given by metric k_{AB}, map ψ_a^A. Kinematics are

$$k_{AB,t} + \hat{v}^j k_{AB,j} = 0,$$
$$\psi^A{}_{i,t} + \left(\hat{v}^j \psi^A{}_j\right)_{,i} = 2\hat{v}^j \psi^A{}_{[i,j]},$$

and dynamics are

$$(\sqrt{\gamma_x}\mathcal{U})_{,t} + \left(\alpha\sqrt{\gamma_x}\mathcal{F}^i\right)_{,i} = \text{source terms}$$

where the conserved $\mathcal{U} = (S_j, \tau)^T$ follow from stress-energy

$$T^{ab} = (e+p)u^a u^b + p\, g^{ab} + \pi^{ab}.$$

Note constraints

$$\psi^A_{[i,j]} = 0.$$

Numerical Methods

All equations hyperbolic PDEs. Two classes:

1. Conservation laws (dynamics, ψ). Standard HRSC methods:
   - Either TVD or WENO reconstruction, HLL Riemann solver, or
   - Lax-Friedrichs flux-split method using WENO reconstruction.
2. Hamilton-Jacobi equations (k_{AB}). Standard methods:
   - ENO or WENO reconstruction;
   - Lax-Friedrichs approximate Hamiltonian.

Complications:

- Constraints not enforced – unnecessary.
- Conversion $\mathcal{U} \to \mathbf{w}$ *very* expensive nonlinear root-find.
- EOS considerably more complex – invent extension of standard.

Wave structure

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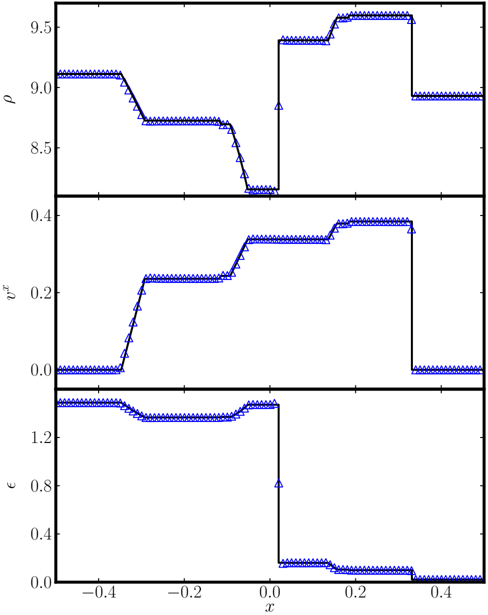
- Wave structure similar to MHD; fast and slow acoustic plus shear waves.
- Newtonian results cleanly separate the waves; 2- and 6-waves only clear in shear.
- SR results: waves cluster (EOS effect). Glitch at (trivial) contact converges.
- Deformation seen in ψ_X^Y.



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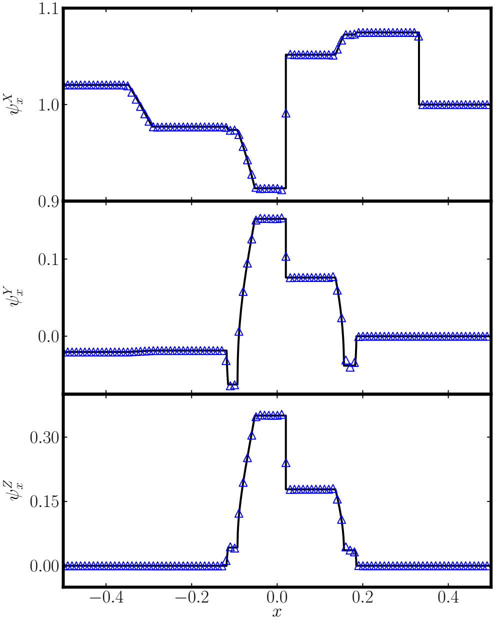

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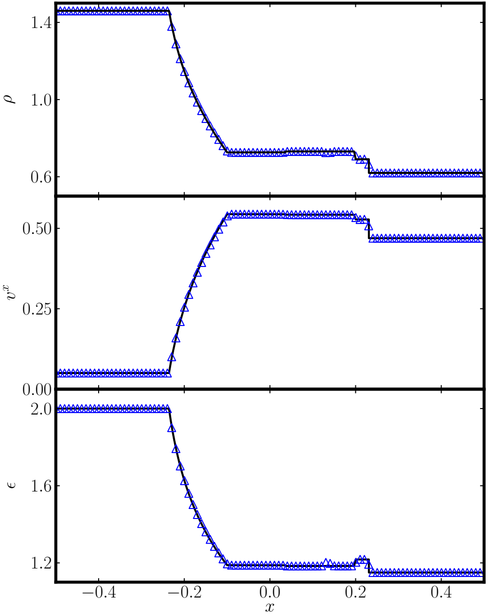

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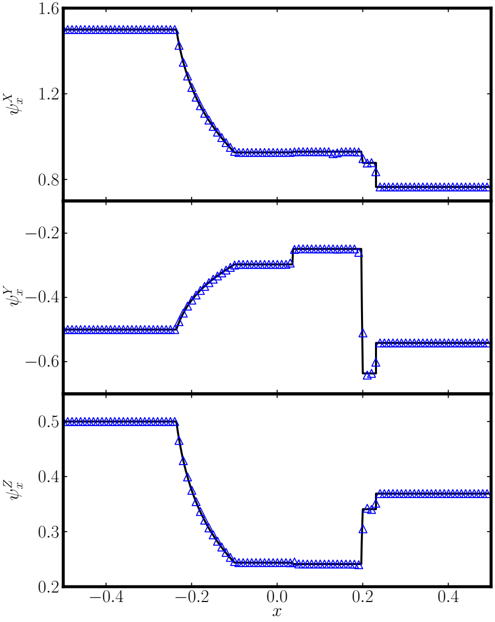
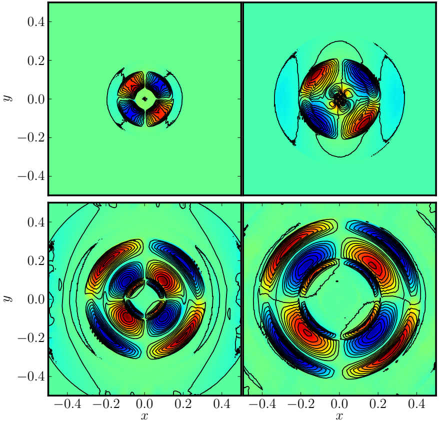


Rotor tests

Newtonian literature suggests problems with naive evolution of ψ:

1. hyperbolicity issues explain this;
2. fixes can be implemented
   1. constraint addition in sources stabilizes it
   2. constraint damping used by some groups.



However, no problem with rotor tests in Newtonian or SR!

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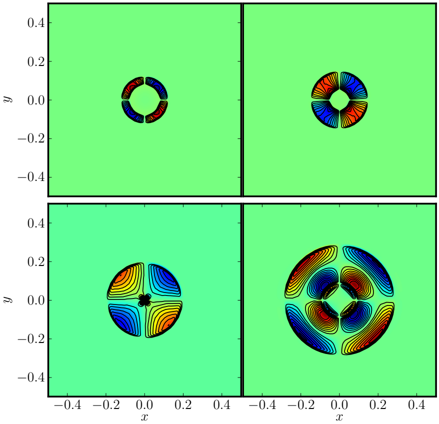
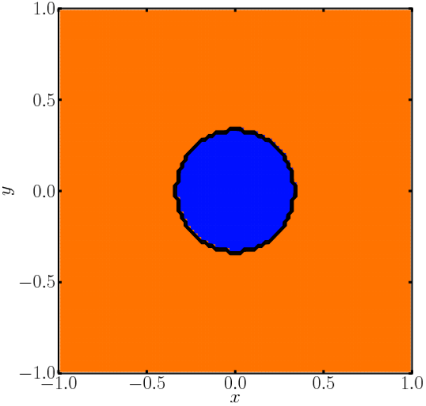


Shattering

- To *shatter* elastic matter need ansatz for
  - fracture
  - energy release
  - configuration change
  - matter space change.
- Proof of principle: instantaneous relaxation:
  - no energy loss
  - configuration and matter space to relaxed state.
- Numerically it "works", but how to do the real case?



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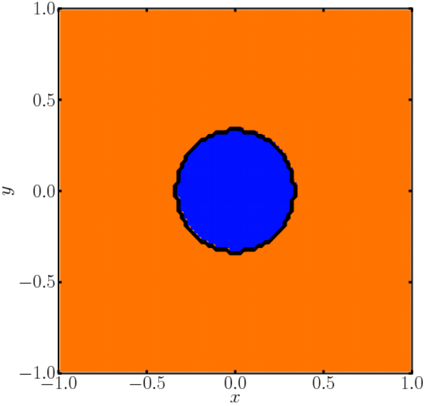

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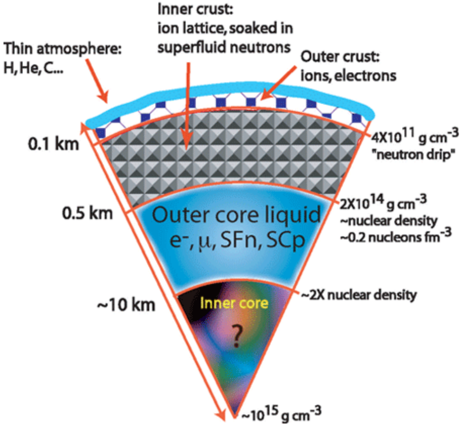
Interfaces

Transition lengths between layers typically $\sim 10\,\text{cm}$.

Practical solution: infinitely thin interfaces.

Level set methods locate interface.

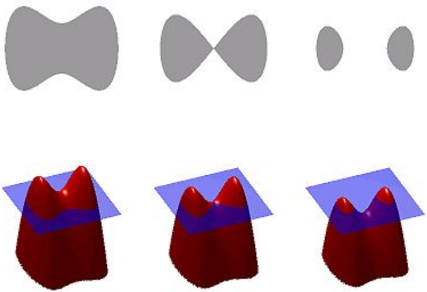
Use ideal GRMHD and elasticity: continuity of traction boundary conditions.



Numerical Methods for interfaces

Capture interface as zero level set of scalar field ϕ:

- Deals with topology change;
- Advected with flow velocity (in ideal case).
- Evolved by Hamilton-Jacobi equation, same as k_{AB}.



Impose boundary condition at interface:

- Continuity of traction;
- Additional conditions for e.g. entropy:
  - Best solution: solve multi-material Riemann Problem.
  - Approximate: extrapolate entropy (Ghost Fluid Method).
- Impose in normal direction; extrapolate as needed.

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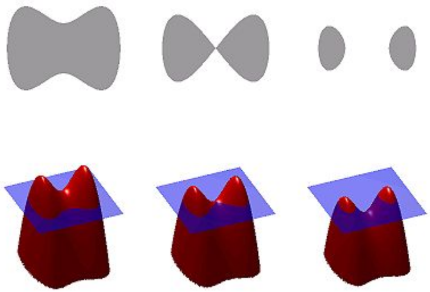

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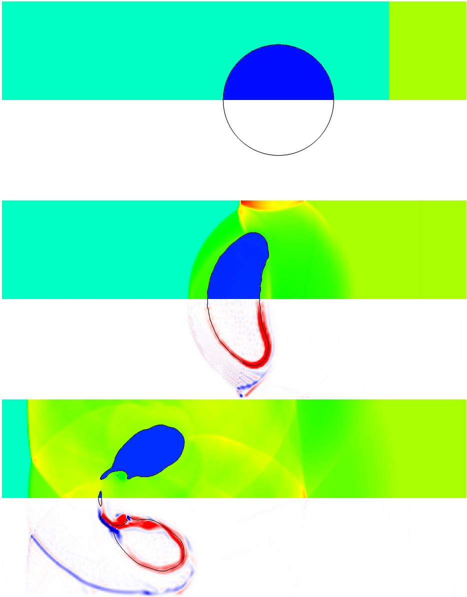
"Bubbles"

In Newtonian hydro a shock deposits vorticity near an interface – curls up.

Relativity compresses the effect without major change.

Small magnetic fields cause splitting - vorticity propagates. Large magnetic fields stop roll-up.

Elasticity also propagates vorticity away.



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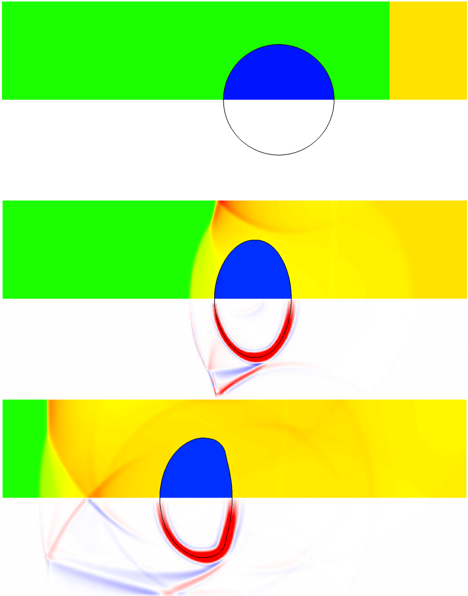

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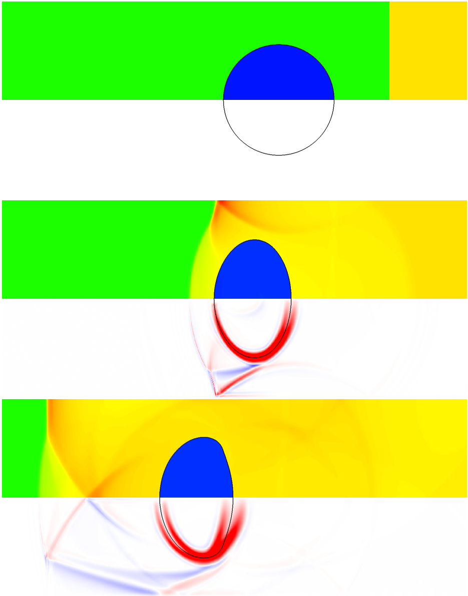

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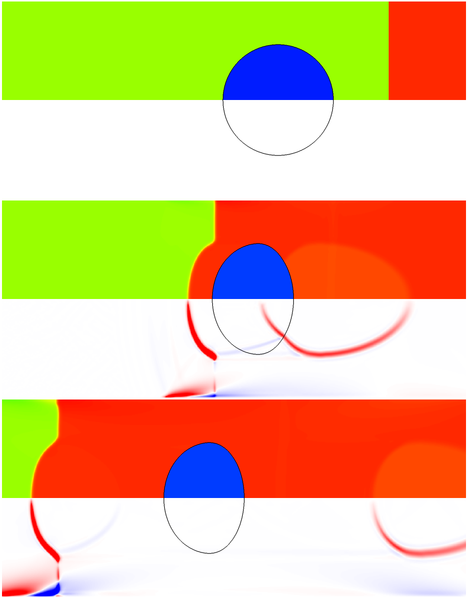

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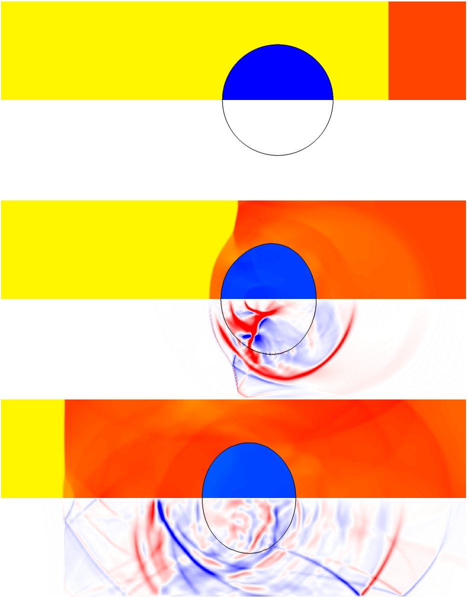

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- Numerical relativistic elastic matter seems straightforward but currently
  - costly
  - not robust.
- Standard interface methods extend to relativity, MHD, elasticity, but currently
  - either expensive, or
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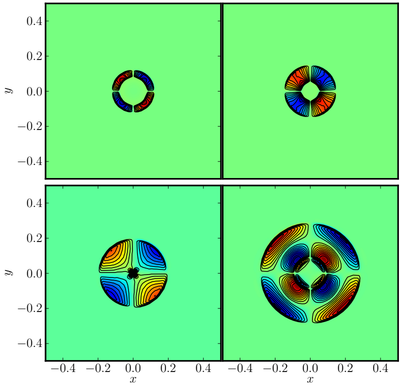
- Vorticity propagation obvious: impact post-merger dynamics unclear.



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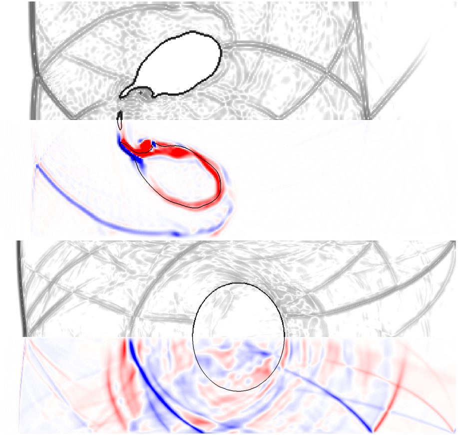

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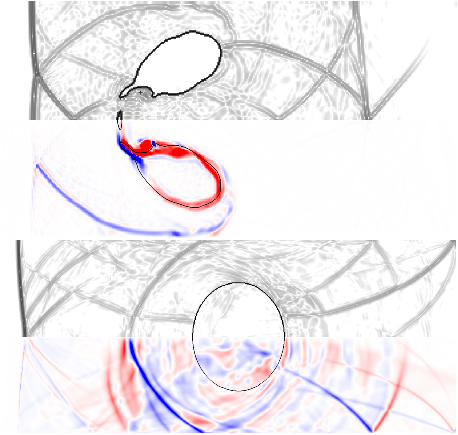

Equations

For completeness we note the full system:

$$k_{AB,t} + \hat{v}^j k_{AB,j} = 0,$$
$$\psi^A{}_{i,t} + \left(\hat{v}^j \psi^A{}_j\right)_{,i} = 2\hat{v}^j \psi^A{}_{[i,j]},$$

and, as given earlier

$$\left(\sqrt{\gamma_x}\mathcal{U}\right)_{,t} + \left(\alpha\sqrt{\gamma_x}\mathcal{F}^i\right)_{,i} = \text{source terms.}$$

with (introducing $\pi = v^i v^j \pi_{ij} = \gamma^{ij}\pi_{ij}$, and ignoring gauge terms)

$$\mathcal{U} = \begin{pmatrix} S_j \\ \tau \end{pmatrix} = \begin{pmatrix} nhW^2 v_j + \pi_{ij} v^i \\ nhW^2 - p - D - \pi \end{pmatrix}, \quad \mathcal{F}^i \sim \begin{pmatrix} nhW^2 v_j \hat{v}^i + p\delta^i{}_j + \pi^i{}_j \\ (nhW^2 - D)\hat{v}^i + \pi^{0i} \end{pmatrix}.$$

We also have constraints

$$\psi^A_{[i,j]} = 0,$$

and an EOS $\epsilon \equiv \epsilon(n, I^1, I^2, s)$ where $n, I^{1,2}$ are scalar invariants of $k^A{}_B$.

EOS

The EOS depends on the strain g^{AB} compared to the reference k_{AB} and e.g. the entropy, in addition to any polarizing effects.

Simplify in two ways:

1. **Homogeneous**: $\epsilon \equiv \epsilon(g^{AB}, k_{AB}, s)$
2. **Isotropic**: $\epsilon \equiv \epsilon(\rho, I^{1,2}, s)$ – the strain dependence is encoded in the invariants of $k^A{}_B$.

Simple tests here use toy EOS using gamma-law fluid plus term proportional to *a* shear scalar,

$$\epsilon = \frac{K(s)}{\gamma - 1} n^{\gamma-1} + \kappa n^{\lambda-1} \mathcal{S}(I^1, I^2).$$

Existence and uniqueness of weak solutions requires EOS restrictions (as yet unclear).

Con2Prim

Converting $(k_{AB}, \psi^{A}{}_{i}, S_j, \tau) \to (v^i, s)$ is the only remaining task.

Standard iterative approach:

1. *Guess* four quantities: $\overline{p - \pi}$ and $\overline{\pi_{ij} v^j}$;
2. Compute all terms consistent with the guess; in particular, $\overline{n}, \overline{h}$ can be found;

$$
\begin{aligned}
D &= \sqrt{\det(k)}\det(\psi), & Z &= \tau + D + \overline{p - \pi}, & \tilde{S}^2 &= \gamma^{ij}\left(S_i - \overline{\pi_{ik} v^k}\right)\left(S_j - \overline{\pi_{jk} v^k}\right), \\
v^2 &= \frac{\tilde{S}^2}{Z^2}, & W &= (1 - v^2)^{-1/2}, & n &= \frac{D}{W}, \\
h &= \frac{Z}{nW^2}, & v_i &= \frac{S_i - \overline{\pi_{ik} v^k}}{Z}, & & \text{rest follows}
\end{aligned}
$$

3. Use the EOS to compute p and π_{ab} from the above;
4. Compute the residuals for the guesses.

Reduces to standard approach for hydro; *very* expensive (50% of computational time).

More complex bubbles

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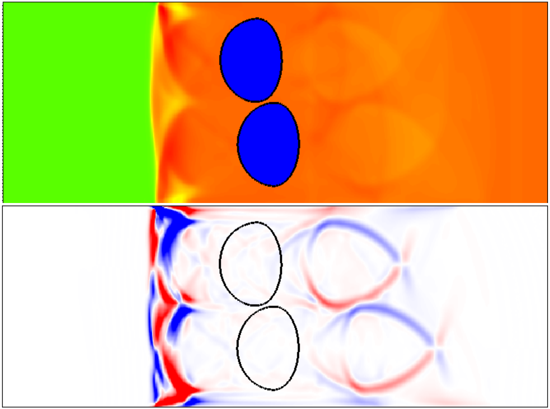
The special geometry of the shock bubble tests is not important.

Even with multiple bubbles in “random” positions the vorticity propagation effects are the same.

These tests are SR MHD at equipartition (plasma $\beta = 1$).



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