GRAVITATIONAL WAVES

Project student lectures 2013

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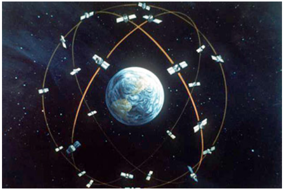
Exercise I

Clocks in GPS satellites

Assume time is measured by laser beams just as in the canonical example of Special Relativity. According to Einstein's Equivalence Principle, being in a gravitational field is no different from travelling in an accelerating rocket. Therefore, clocks are ticking at different rates depending on their location in the gravitational field.

Are the clocks in GPS satellites ticking faster or slower compared to the surface of the earth?



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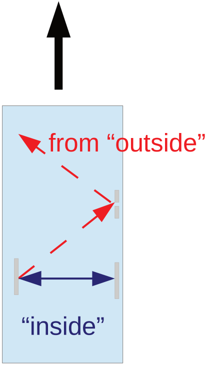

Simple (heuristic) explanation

1. Just like in Special Relativity, fast moving (accelerating) rockets appear to have slower ticking clocks.

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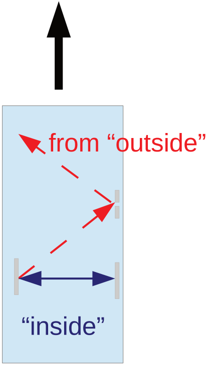

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2. Clocks on earth and GPS satellites feel gravitational field of the earth $\Rightarrow$ same effects as in accelerating rocket (Einstein's Equivalence Principle)

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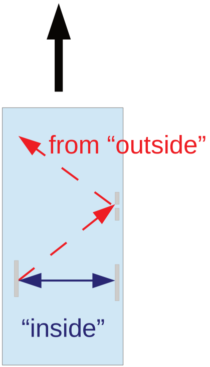

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2. Clocks on earth and GPS satellites feel gravitational field of the earth $\Rightarrow$ same effects as in accelerating rocket (Einstein's Equivalence Principle)
3. The acceleration is greater on the surface $\Rightarrow$ earth-based clocks tick slower, or conversely, clocks in GPS satellites *tick faster*

Exercise II

Metric transformations

The flat space metric in Cartesian coordinates is the 3×3 identity matrix. How does the same metric look in cylindrical and spherical coordinates?

Can you think of a coordinate system in which the flat space metric is *not* diagonal?

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Polar: $ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$
Spherical: $ds^2 = dr^2 + r^2 \sin^2 \vartheta\, d\varphi^2 + r^2\, d\vartheta^2$

Matrix form

Cartesian	Polar	Spherical
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \vartheta & 0 \\ 0 & 0 & r^2 \end{pmatrix}$

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Example of non-diagonal metric: $x = \xi + \eta, \; y = \eta + \kappa, \; z = \kappa$

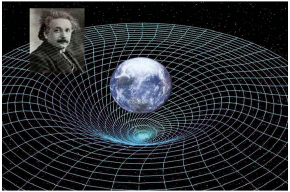
$$\Rightarrow \; \boldsymbol{g} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Recap

Einstein's Equations

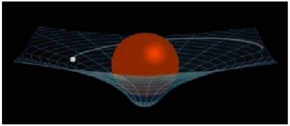
- Matter and energy warp spacetime

$$\underbrace{R_{ab} - \frac{1}{2} R\, g_{ab}}_{curvature} = \underbrace{8\pi T_{ab}}_{energy/momentum}$$



Geodesics

- Free particles move along *geodesics*, the equivalent of a straight line in curved spacetime
- Geodesics are the shortest paths between two points.



Solutions to Einstein's Equations

Solving the equations is hard . . .

- Set of 10 coupled non-linear, second-order, partial differential equations for the metric g_{ab}
- Coordinates (gauge) is not specified
- Some exact solutions are known, but mostly for "simple" cases (time-independent, symmetric)
  - Minkowski metric: flat spacetime
  - Schwarzschild: unique spherical symmetric vacuum solution outside of a massive body (non-rotating black hole)
  - Kerr solution for a rotating black hole
  - Friedmann-Robertson-Walker cosmological solution

Weak field limit

Assumptions

- Consider a weak gravitational field
- There exist global coordinates such that the metric is "almost" flat, i.e.

$$g_{ab} = \eta_{ab} + h_{ab} \; , \qquad \text{with } |h_{ab}| \ll 1$$

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Strategy

- Use this ansatz and calculate the curvature
- Keep terms up to $\mathcal{O}(h_{ab})$
- Coordinates are not arbitrary any more (we need $|h_{ab}| \ll 1$), but there is still some freedom

$\Rightarrow$ Use transverse-traceless Loren(t)z gauge to simplify equations

Einstein's Equations

$$R_{ab} - \frac{1}{2} R\, g_{ab} = 8\pi T_{ab}$$

$$\Rightarrow \quad \partial^c \partial_c h_{ab} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) h_{ab} = -16\pi T_{ab}$$

Solution of the wave equation

- In vacuum, where $T_{ab} = 0$, we find the plane wave solution
$$h_{ab} = A_{ab}\, e^{i\, k_c x^c}$$
with the following properties
    - $\eta_{ab} k^a k^b = 0$: wave travels at the speed of light
    - $A_{ab} k^a = 0$: wave is transverse
    - $\eta_{ab} A^{ab} = 0$: solution is trace-free
    - $A_{0i} = 0$ for $i = 1, 2, 3$

Wave equation

Explicit form

- Consider a wave travelling in z-direction

$\Rightarrow$ $k_a = (-\omega, 0, 0, \omega)$

- Only **2** independent functions remain

$$h_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ds^2 = -dt^2 + (1 + h_+)\, dx^2 + (1 - h_+)\, dy^2 + 2h_\times\, dx\, dy + dz^2$$

Effect of a GW

The metric changes slightly, so what?

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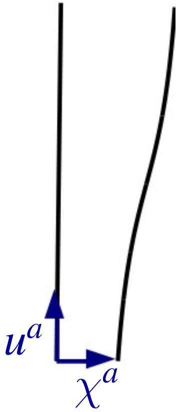
Geodesic deviation

- Separation of free falling particles

$$\nabla_u \nabla_u \chi^a = R^a{}_{bcd}\, u^b u^c \chi^d$$

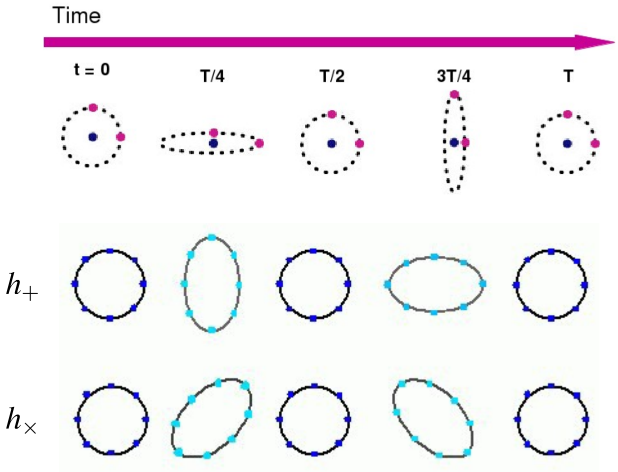
$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}\chi^a = R^a{}_{00d}\chi^d = \frac{1}{2}\frac{\partial^2 h_{ad}}{\partial t^2}\chi^d$$

- Proper distance between particles changes



Visualization

Consider a GW travelling into the screen.



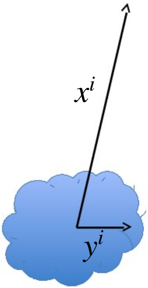
Generation of Gravitational Waves

General solution

$$\partial^c \partial_c h_{ab} = -16\pi T_{ab}$$
$$\Rightarrow \quad h_{ab}(t, x^i) = 4 \int \frac{T_{ab}(t - |x^i - y^i|, y^i)}{|x^i - y^i|} d^3 y$$

Strategy

- Assume far field, $\|x^i\| \gg \|y^i\|$
- Expand in leading order terms
- Use transverse, trace-free gauge and conservation law $\partial_a T^{ab} = 0$.



The quadrupole formula

After a good bit of calculation …

$$h_{ij}(t,x) = \frac{2}{r}\frac{\partial^2}{\partial t^2} \underbrace{\int y^i y^j \, T^{00}(t-r, y^i)\, d^3y}_{\text{mass quadrupole moment in TT gauge}}$$

$(r = \|x^i\|)$

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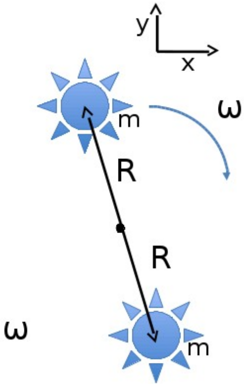
Interpretation

- The gravitational wave is caused by an accelerating mass quadrupole moment
- There is no monopole or dipole radiation. (Follows from the conservation of energy and momentum.)

Example

Binary on circular orbit

$$h_{ij} = \frac{2}{r}\frac{\partial^2}{\partial t^2}\left(2m\, y^i y^j\right)$$



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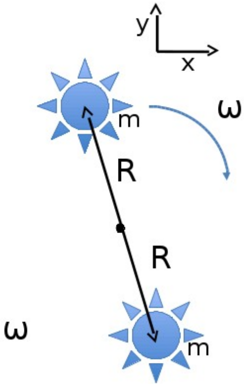
$$h_{ij} = \frac{2}{r}\frac{\partial^2}{\partial t^2}\left(2m\, y^i y^j\right)$$

$$\Rightarrow \quad h_+ = -\frac{8m\omega^2 R^2}{r}\cos(2\omega t)$$

$$h_\times = -\frac{8m\omega^2 R^2}{r}\sin(2\omega t)$$



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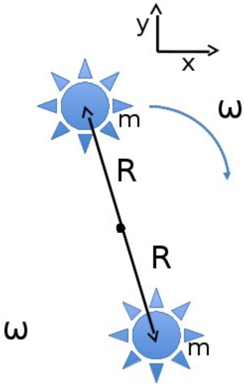

Home assignment

What is the GW strain on earth produced by a neutron-star binary that is 100 Mpc away and separated by 50km?

Energy

Few words about the energy carried by GW

- Local energy cannot be defined strictly in GR
- In linearised gravity, use flat background metric spacetime to obtain energy in GW as

$$T_{ab}^{(\mathrm{GW})} = \frac{1}{32\pi} \left(\partial_a h_{cd}\right) \left(\partial_b h^{cd}\right)$$

- Combined with quadrupole formula and integrated over the sphere yields the total luminosity of the source

$$\mathcal{L} = \frac{1}{5} \frac{\partial^3 Q^{ij}}{\partial t^3} \frac{\partial^3 Q_{ij}}{\partial t^3}$$

where Q_{ij} is the mass quadrupole moment.

Exercise IV

Chirp Signal

An equal-mass binary on circular orbits with velocities v and masses m is characterized by the energy

$$E(t) = -m\,v(t)^2$$

and the gravitational-wave flux

$$\mathcal{L}(t) = \frac{2048}{5} v(t)^{10}\ .$$

Determine from the energy-balance law

$$\frac{dE(t)}{dt} = -\mathcal{L}(t)$$

the evolution of v. How does the GW signal (h_+) look like?
(Hint: Keplerian motion: $m\omega = 4v^3$.)