Gravitational-Wave Detectors

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Goal and Lecture Outline

Goal: Get a basic understanding of the main noise sources in an interferometer like LIGO, and how we can hope to detect GWs.

Outline of Lecture: We'll cover

- Basics of interferometric GW detectors.
  - Recap of GWs.
  - $h(t)$ and phase shift in a basic interferometer
- Main noise sources, their size, how they're controlled.
  - Thermal noise
  - Optical readout noise: shot noise, radiation pressure noise, & the standard quantum limit
  - Seismic noise

Gravitational-Wave Detection for Theorists

Wave Equation for Linearized Gravity

- A concrete treatment of GWs is based on linearized perturbations around a fixed background metric in general relativity.
- The perturbation obeys a wave equation. In flat space:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})h_{ab} = -\frac{16\pi G}{c^4}T_{ab}$$

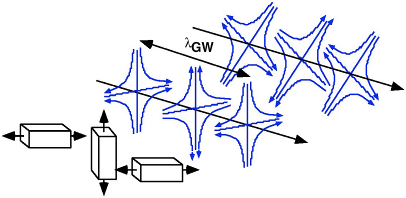
wave operator

perturbation (GW)

coupling constant (10^{-42}/N)

source stress tensor (density of mass & energy)

- Solution: Transverse waves with 2 polarizations rotated by 45°.



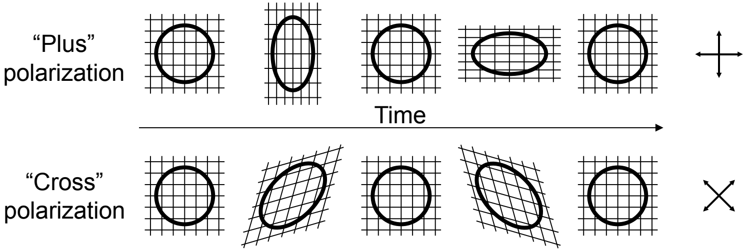
Example

General solution for a GW propagating in the z direction in empty space ($T_{ab} = 0$):

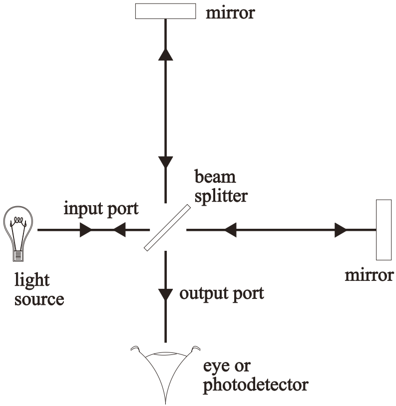
$$h_{ab}(t,z) = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} h_+(ct-z)}_{\text{“plus” polarization}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} h_\times(ct-z)}_{\text{“cross” polarization}} .$$

- h_+ and $h_\times$ are arbitrary functions.
- No component in the t direction or the direction of propagation (z direction) — GWs are transverse.
- GWs propagate at the speed of light.

Gravitational waves are deformations of space itself, stretching it first in one direction, then in the perpendicular direction.

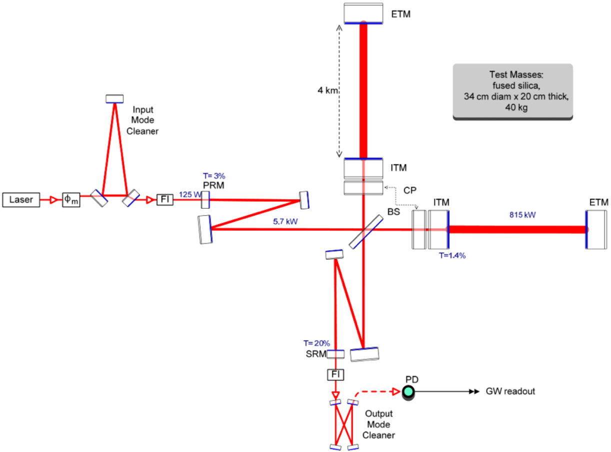


Interferometric GW Detectors (simplest case)



- Take advantage of the tidal nature of GWs.
- The simplest design is just a Michelson interferometer. A laser is used to measure the relative lengths of two orthogonal paths (“arms”).
- As a GW passes, it changes the path length along the two arms, changing the interference pattern at the output photodetector.

Interferometric GW Detectors (more sophisticated)



- Add partially reflecting mirrors – turns arms into Fabry-Perot cavities, effectively increasing length by $\times$30–50.

- Add power recycling mirror – returns spent light to interferometer. Effectively increases laser power by $\times$20.

Current Ground-based detectors

- A number second-generation “advanced” detectors are under construction:
  - Two LIGO detectors in the US (one in Hanford, WA with 4 km arms, and one in Livingston, LA, with 4 km arms).
  - The Virgo detector in Pisa, Italy (with 3 km arms).
  - The KAGRA detector in Kamioka, Japan (with 3 km arms).

- The first-generation GEO-600 detector in Hannover, Germany (with 600 m arms) is also currently operating.

LIGO Hanford Observatory

- For two free particles initially at rest on the x-axis separated by the coordinate distance L_*, the proper distance between the particles changes under the passage of a plane gravitational wave propagating in the z-direction according to:

$$L(t) = \int_0^{L_*} \sqrt{ds^2} = \int_0^{L_*} dx \sqrt{1 + h_+(t)} \approx \left(1 + \frac{1}{2} h_+(t)\right) L_* \,. \tag{1}$$

- The fractional displacement is thus

$$\frac{\Delta L(t)}{L_*} := \frac{L(t) - L_*}{L_*} = \frac{1}{2} h_+(t) \tag{2}$$

Detecting GWs with a laser interferometer

- Consider a $+$-polarized plane wave incident on a laser interferometer directly from above, with x- and y-axes directed along the interferometer's arms. Then for laser light traveling in the x-direction:

$$0 = ds^2 = -dt^2 + [1 + h_+(t)]\, dx^2 \tag{3}$$

The light travel time down the x-arm is thus

$$T_{\mathrm{out,x}} = L_{x*} + \frac{1}{2} \int_0^{L_{x*}} h_+(t)\, dx \tag{4}$$

where L_{x*} is the unperturbed length of the arm, and where we have ignored second-order terms in h_+.

- Provided the period of the GW is much greater than the light travel time, then h is approximately constant over the round-trip. Then

$$T_{\mathrm{out,x}} = L_{x*} + \frac{1}{2} h_+(t)\, L_{x*} \tag{5}$$

- The total round-trip time is

$$T_{\mathrm{tot,x}} = 2L_{x*} + h_+(t)\, L_{x*} \tag{6}$$

- Similarly, for the y-arm:

$$T_{\mathrm{tot,y}} = 2L_{y*} - h_+(t)\, L_{y*} \tag{7}$$

- Thus,

$$\Delta T \equiv T_{\mathrm{tot,x}} - T_{\mathrm{tot,y}} = 2L_* \, h_+(t)\,, \tag{8}$$

where in the last equality we assumed that $L_{x*} = L_{y*} =: L_*$.

Phase shift

- The phase shift in the recombined laser light is

$$\Delta\phi := 2\pi f_{\mathrm{laser}} \Delta T = \frac{4\pi\, L_*\, h_+(t)}{\lambda_{\mathrm{laser}}} \,. \tag{9}$$

- Substituting $\lambda_{\mathrm{laser}} = 10^{-6}\mathrm{m}$, $L_* = 4\ \mathrm{km}$, and $h_+(t) \sim 10^{-21}$ (which is the strength of GWs that LIGO hopes to detect), we get

$$\Delta L(t) = 2 \times 10^{-18}\mathrm{m}\,, \tag{10}$$

$$\Delta\phi \sim 3 \times 10^{-9}\,. \tag{11}$$

where $\Delta\phi$ includes a factor of 50 for the effective increase in arm length due to the use of Fabry-Perot cavities.

Reality check

Exercise: Show that it is impossible for LIGO to detect GWs.

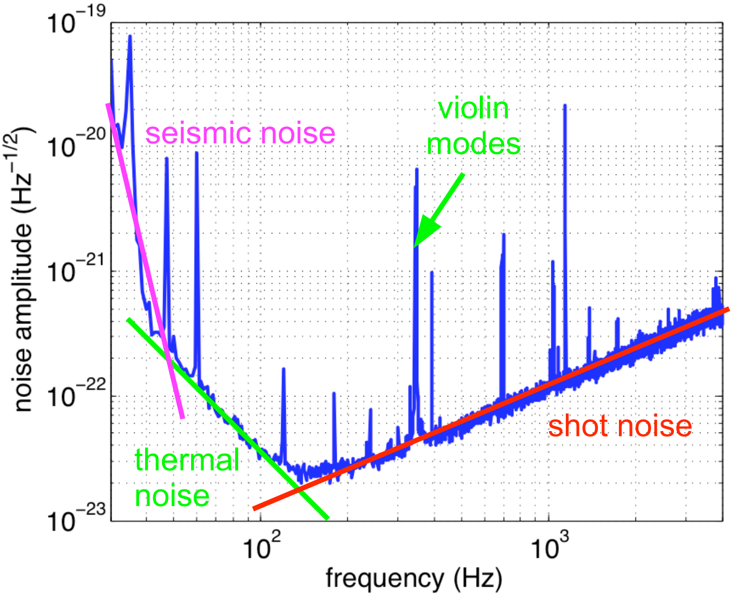
Answer: Maximum expected GW amplitude is about $h < 10^{-21}$, so the change in arm lengths is $\sim 10^{-18}\mathrm{m}$ and the phase shift is $\Delta\phi \sim 10^{-9}$. These are:

- 10^{10} times smaller than the phase shift needed to move one fringe in a Michelson-type experiment.
- 10^{12} times smaller than the laser wavelength, $\sim 1\mu\mathrm{m}$ (the size of our "ruler").
- 10^{9} times smaller than the size of the atoms in the mirror whose position we are trying to measure (and which are oscillating due to thermal noise).
- 10^{12} times smaller than the typical motion of the ground due to seismic noise (speeds $\sim 1\mu\mathrm{m/s}$).
- Next: Consider each of these in turn, and see how they're handled.

Noise Sources & Mitigation

This material is taken from the book *Fundamentals of Interferometric Gravitational-Wave Detectors*, by Peter Saulson.

LIGO-Hanford Detector Noise Spectrum, 2007



Thermal Noise

- Equipartition Theorem: each degree of freedom of a system in thermodynamic equilibrium at temperature T should have an energy whose expectation value is $\frac{1}{2}k_B T$.

- Main physical manifestations of thermal noise:
  - Random motion of atoms in mirrors.
  - Vibration in wire suspensions of mirrors ("violin modes").
  - Swinging of the mirror pendula.

- Induces random "jittering" of positions of mirrors.
  - E.g., $m = 10kg$ mirror suspended from a $\ell = 1m$ wire:

$$\Delta L_{\rm rms} \simeq \sqrt{\frac{k_B T \ell}{mg}} \simeq 6 \times 10^{-12} m\,.$$

  - About 10^7 times larger than our GW!

Fluctuation-Dissipation Theorem

Applies to any linear system in thermodynamic equilibrium.

- For a system with impedance $Z \equiv F_{ext}/v$ the power spectrum of the system's random fluctuating motion is

$$S_{\rm therm}(f) = \frac{k_B T}{\pi^2 f^2}\,{\rm Re}(Z^{-1}(f))\,.$$

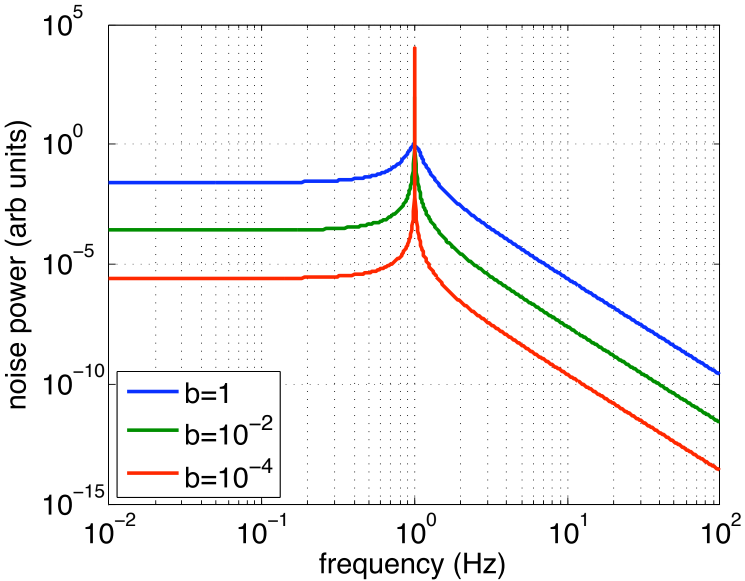
- E.g., for harmonic oscillator with mass m, spring constant k, and dissipative force $-bv$:

$$S_{\rm therm}(f) = \frac{k_B T}{\pi^2}\frac{b}{b^2 f^2 + (2\pi m)^2[f^2 - f_0^2]^2}\,.$$

  Here $f_0 = \sqrt{k/m}/2\pi$ is the resonant frequency.

- Can show total noise power (or RMS displacement) does not depend on the magnitude b of the dissipation, but the shape of the power spectrum does depend on b.

Fluctuation-Dissipation Theorem



Fluctuation-Dissipation Theorem

- Can minimize impact by making dissipation (b) small. Concentrates noise power at resonant frequency f_0.
  - Suspend mirrors as pendulums (low b, resonant frequency $\sim 1 Hz$ – well below observation band).
  - Use high-quality-factor wires for suspensions, so vibrational motion of "violin modes" is concentrated in narrow frequency range.
  - Minimize the thermal noise from internal vibrations of the mirrors by making them from material with very low dissipation at acoustic frequencies at room temperature (fused silica).

Optical Readout Noise, pt. 1: Shot Noise

- Want to measure changes in phase at the level $\Delta\phi \sim 10^{-11}$. What is the limit?
- Measure phase from power received at photodiode. Light comes in discrete packets (photons) with random arrival times. Quantum fluctuations in number of photons arriving in some time interval limit how accurately we can measure power or phase.
- Rate of photons at photodiode:

$$\langle n \rangle = \frac{\lambda}{2\pi\hbar c} P_{\mathrm{out}} \,.$$

- Output power :

$$P_{\mathrm{out}} = P_{\mathrm{in}} \cos^2([L_x - L_y] 2\pi/\lambda) \,.$$

Optical Readout Noise, pt. 1: Shot Noise

- For operating on "half fringe" $P_{\rm out} = \frac{1}{2} P_{\rm in}$, power/photon fluctuations in time τ equivalent to length changes via

$$\frac{\Delta L}{L} = \frac{\Delta P_{\rm out}}{L} \frac{dL}{dP_{\rm out}} = \frac{\lambda}{4\pi L} \frac{\Delta n}{\langle n \rangle} = \frac{1}{L} \sqrt{\frac{\lambda \hbar c}{4\pi P_{\rm in} \tau}} \, .$$

  where we use $\Delta n \simeq \langle n \rangle^{1/2}$ for a Poisson process.

- Corresponding spectral density of this photon shot noise is white (independent of frequency) with magnitude

$$S_{\rm shot} = \frac{1}{L^2} \frac{\lambda \hbar c}{2\pi P_{\rm in}} \, .$$

  - Want highest laser power $P_{\rm in}$ possible to reduce shot noise.

Optical Readout Noise, pt. 2: Radiation Pressure

- Can't increase laser power indefinitely without repercussions.
  - Photons bouncing off mirrors impart momentum, moving the mirrors stochastically.
  - More power $\longrightarrow$ more photons $\longrightarrow$ more radiation pressure noise.
- Force on mirror due to reflection of EM wave of power P is

$$F = P/c\,.$$

- Spectrum of radiation pressure force:

$$F(f) = \sqrt{\frac{2\pi\hbar P_{\rm in}}{c\lambda}}\,.$$

Optical Readout Noise, pt. 2: Radiation Pressure

- Corresponding motion of each mirror (from $F = ma$):

$$x(f) = \frac{F(f)}{m(2\pi f)^2}\,.$$

- The power spectrum of the fractional relative length change from the sum of the 2 arms is then

$$S_{\rm rad}(f) = \left(\frac{2x(f)}{L}\right)^2 = \frac{1}{m^2 L^2 f^4}\,\frac{\hbar P_{\rm in}}{2\pi^3 c\lambda}\,.$$

  - Want lowest laser power $P_{\rm in}$ possible to reduce radiation pressure noise.

Optical Readout Noise, pt. 3: Standard Quantum Limit

- Shot noise decreases with laser power, while radiation pressure increases with laser power.
- At any given frequency f the sum $S_{\rm shot} + S_{\rm rad}$ is minimized by choosing the power $P_{\rm in}$ to make $S_{\rm shot} = S_{\rm rad}$. The total noise is then

$$S_{\rm SQL} = \frac{1}{(\pi f L)^2} \frac{\hbar}{m} \, .$$

- This minimum noise level is called the standard quantum limit.
  - Though derived using details of the readout scheme, it is a direct result of the Heisenberg uncertainty principle.

Optical Readout Noise, pt. 3: Standard Quantum Limit

- E.g.: For LIGO $P_{\rm in} \sim 10\rm W$, $\lambda = 1\mu\rm m$, $m \sim 10\rm kg$, so at $f = 100\rm Hz$ we could get noise as low as

$$S_{\rm SQL} \sim 6 \times 10^{-48}/\rm Hz\,.$$

The corresponding RMS length fluctuations are approximately

$$\Delta L_{\rm RMS} \simeq L\sqrt{f\, S_{\rm SQL}} \sim 10^{-19}\rm m\,.$$

This is indeed lower than our target of $10^{-18}\rm m$ – though not by much!

- Advanced detectors will need to use tricks such as "squeezed light" to beat the SQL by shifting quantum uncertainty into degrees of freedom not being measured (not related to the relative length of the arms).

Seismic Noise

- Seismic noise (shaking of the ground, due to earthquakes, weather, human activity, etc.) is not a fundamental noise source like thermal noise or quantum uncertainty, but it pretty important for detectors on the Earth.

- Typical level: for mirrors separated by a large distance L the relative distance fluctuations $\Delta L/L$ have power spectrum

$$
\begin{aligned}
S_{\mathrm{seismic}} &\simeq \frac{10^{-18}\mathrm{m}^2\,\mathrm{Hz}^{-1}}{L^2}\left(\frac{10\mathrm{Hz}}{f}\right)^4, \quad f > 10\,\mathrm{Hz} \\
&\simeq 10^{-29}\,\mathrm{Hz}^{-1}\left(\frac{100\mathrm{Hz}}{f}\right)^4, \quad (L = 4\mathrm{km})
\end{aligned}
$$

  - This is about 10^{18} times the SQL we're trying to reach!

Seismic Isolation

- Need to suppress length fluctuations by about 10^{10} in amplitude (10^{20} in power).
- Recall: Mirrors are suspended as pendula (simple harmonic oscillators). Coupling between mirror position x_m and ground position x_g is

$$mx_m'' = -k(x_m - x_g)\,.$$

- In frequency domain:

$$\frac{x_m}{x_g} = \frac{f_0^2}{f_0^2 - f^2} \propto f^{-2} \qquad f \gg f_0$$

Mirror is passively isolated from ground motion at high frequencies ($f \gg f_0 \sim 1\mathrm{Hz}$).

Seismic Isolation

- Increase isolation from ground by "stacking":
  - Suspend mirror pendulum from another pendulum.
  - Suspend mirror pendulum from another SHO like a spring, itself suspended from another spring, etc.
  - For N layers of the stack, $x_m/x_g \to (f_0/f)^{2N}$.

- LIGO also uses active isolation: sensors monitor motion of suspension system; control motors that push back on system to cancel ground motion.
  - Can get required 10^{10} amplitude factor, though not easy.