PX4224: Advanced General Relativity and Gravitational Waves

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Lecture 1: Curved Geometry and Geodesics [Hartle, Ch. 7 & 8]

The Strong Principle of Equivalence

- **Strong Principle of Equivalence**: In an arbitrary gravitational field, at any given spacetime point, we can choose a locally inertial reference frame such that, in a sufficiently small region surrounding that point, all physical laws take the same form they would take in absence of gravity, namely the form prescribed by Special Relativity.

- Nature has no preferred coordinate system $\Rightarrow$ **we need to be able to write physical laws in a form that is valid in any coordinate system.**

Coordinates

- We must abandon global inertial coordinates (e.g., (t, x, y, z) in flat spacetime): they do not exist for a general curved spacetime. Inertial coordinates exist only locally.
- **Axiom used by Gauss for non-Euclidean geometries**: At any given point in space, there exist a locally Euclidean reference frame such that, in a sufficiently small region surrounding that point, the distance between two points is given by the law of Pythagoras.
- One needs, therefore, to consider general **curvilinear** coordinates x^α.

Line element

- Line element in a local inertial frame: $ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu$, where $\eta_{\alpha\beta} = diag(-1, 1, 1, 1)$ is the flat space metric.

- Switch to a generic frame where coordinates are labelled $x^\alpha(\xi^\alpha)$. The curved spacetime line element is

$$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu = g_{\mu\nu}(\mathbf{x}) dx^\mu dx^\nu \qquad (1.1)$$

- The spacetime metric components $g_{\alpha\beta}$ are symmetric with respect to interchange of α and β. How many independent components do we have in spacetime? *Answer:* 10

Local Inertial Frames

- Inertial reference frames exist only **locally**. Given any event P in spacetime, it is always possible to find coordinates **local inertial coordinates** ξ^α such that

$$g_{\alpha\beta}\big|_P = \eta_{\alpha\beta}\,, \quad \left.\frac{\partial g_{\alpha\beta}}{\partial \xi^\gamma}\right|_P = 0 \tag{1.2}$$

(Hint: higher-order derivatives will play a role later.)

- Local flatness is a consequence of the equivalence principle (a freely falling observer should be in a local inertial frame).

Light Cones & World Lines

- General relativity inherits locally the light cone structure of special relativity
- Points separated from P by infinitesimal coordinate intervals dx^α can be
  1. timelike separated from P ($ds^2 < 0$)
  2. null separated from P ($ds^2 = 0$)
  3. spacelike separated from P ($ds^2 > 0$)
- Light still travels along null world lines, but such lines are no longer necessarily at 45° in a spacetime diagram.
- Particles move along timelike world lines

Vectors in Curved Spacetime

- They are defined locally as tangents to curves passing through an event P; i.e., they are defined in the tangent space at P.
- Vectors at different points "live" in different tangent spaces and are not directly comparable $(\mathbf{a}(P) + \mathbf{b}(Q) = ?)$.
- Forget position vectors: they are not defined locally.
- Displacements $dx^\alpha = x^\alpha_P - x^\alpha_Q$ between infinitesimally separated points P and Q are still vectors.

If we restrict to locally defined vectors, most SR machinery carries over to curved space with $\eta_{\alpha\beta} \to g_{\alpha\beta}$; e.g., the inner product of two vectors is defined as $\mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta$

Bases

A vector at a point can be decomposed in terms of a set of basis vectors. There are two types of bases we will consider:

(i) A **coordinate basis**, denoted $\{\mathbf{e}_\alpha\}$. Pick $\mathbf{e}_\alpha$ as the tangent vector pointing along the x^α coordinate axis:

$$(\mathbf{e}_\alpha)^\beta = \delta^\beta_\alpha \qquad (\text{in } x \text{ coordinates}) \,. \tag{1.3}$$

The coordinate basis vectors are not necessarily unit vectors or orthogonal to one another:

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} \,. \tag{1.4}$$

A coordinate basis is used for most calculations.

Bases

(ii) An **orthonormal basis**, denoted $\{\mathbf{e}_{\hat{\alpha}}\}$. Since spacetime is locally flat, we can always choose coordinates x' near an event P so that $g_{\alpha\beta}(x') \to \eta_{\alpha\beta}$:

$$ds^2 = \eta_{\alpha\beta}\, dx'^{\alpha} dx'^{\beta} \,. \tag{1.5}$$

Choose $\{\mathbf{e}_{\hat{\alpha}}\}$ as the coordinate basis vectors in this special coordinate system:

$$(\mathbf{e}_{\hat{\alpha}})^{\hat{\beta}} = \delta^{\hat{\beta}}_{\hat{\alpha}} \qquad (\text{in } x' \text{ coordinates})\,. \tag{1.6}$$

By construction, the $\{\mathbf{e}_{\hat{\alpha}}\}$ form an orthonormal set:

$$\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} = \mathrm{diag}(-1, 1, 1, 1) \tag{1.7}$$

An orthonormal basis is useful for interpreting measurements done by observers in spacetime.

Hypersurfaces

A 3-dimensional **hypersurface** in spacetime is defined by a condition $f(x^\alpha) = 0$. Examples:

1. $t - t_0 = 0$ (i.e., a $t = \mathrm{const}$ surface)
2. $-t^2 + r^2 + a^2 = 0$ (i.e., a Lorentz hyperboloid in flat spacetime)

Often useful to split a region of spacetime into a "stack" (foliation) of 3D hypersurfaces.

- Each sheet can be considered a surface of constant time in some coordinate system.
- How we define simultaneity in GR.
- Different families of hypersurfaces $\rightarrow$ different choices of time coordinate.

Hypersurfaces

- **Tangent** vectors $\mathbf{t}$ to a hypersurface $f(x^\alpha) = 0$ satisfy

$$t^\alpha \frac{df}{dx^\alpha} = 0 \ . \qquad (1.8)$$

- The **normal** vector n^α is defined by

$$0 = \mathbf{n} \cdot \mathbf{t} = g_{\alpha\beta}\, n^\alpha\, t^\beta \qquad (1.9)$$

  for any tangent vector (there are 3 independent ones).
  - A 3-d hypersurface is said to be **spacelike** if $\mathbf{n} \cdot \mathbf{n} < 0$.
  - A 3-d hypersurface is said to be **timelike** if $\mathbf{n} \cdot \mathbf{n} > 0$.
  - A 3-d hypersurface is said to be **null** if $\mathbf{n} \cdot \mathbf{n} = 0$.
- The 3-d line element on a 3-d hypersurface is obtained by restricting the 4-d line element to the surface.

Geodesics

- Free particles in curved spacetime move along geodesics, which are paths that extremise the proper time, τ:

$$S[x^\mu(\sigma)] \equiv \int_A^B d\tau = \int_0^1 d\sigma \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\,. \qquad (1.10)$$

- The geodesic equations in a general curved spacetime are the Euler-Lagrange equations for the Lagrangian

$$L\left(x^\alpha(\sigma), \frac{dx^\alpha(\sigma)}{d\sigma}, \sigma\right) := \frac{d\tau}{d\sigma} = \sqrt{-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} \qquad (1.11)$$

Geodesics

- The geodesic equation is:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \qquad (1.12)$$

where

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}\left(\frac{\partial g_{\mu\beta}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu}\right) \qquad (1.13)$$

are the Christoffel symbols, and $g^{\alpha\beta}$ are the components of the inverse matrix to $g_{\alpha\beta}$.

- Note that $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$
- How many independent Christoffel symbols are there?
Answer: 40 in 4-d; 18 in 3-d; 6 in in 2-d.

Null & Spacelike Geodesics

- Because $d\tau^2 = -ds^2 = 0$ along any null curve, one cannot use proper time as a parameter along light rays
- However, it can be shown that one can always choose an affine parameter λ such that the same form of Eq. (1.12) holds for **null geodesics**:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \qquad (1.14)$$

- For **spacelike geodesics** we can use the same equation again with the proper length σ as the parameter.

Exercise 1

The line-element of the unit 2-sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

(a) Calculate the Christoffel symbols associated to this metric. Since it's two dimensional, there are only six.

(b) A particle is moving along the equator. What are its initial position x^α and velocity $\frac{dx^\alpha}{d\sigma}$?

(c) Using the geodesic equation, show that the acceleration $\frac{d^2x^\alpha}{d\sigma^2}$ vanishes. What does this tell you?

(d) What measurements could you make to demonstrate the this surface is curved?

Lectures 2 - 4: Gravitational Waves [Hartle, Ch. 16]

What are Gravitational Waves?

- Mass produces spacetime curvature.
- Special relativity imposes a finite speed limit (c) for any type of communication.
- Mass in motion produces ripples in spacetime curvature that propagate at the speed of light: gravitational waves.
- Gravitational waves carry significant amounts of energy. However, due to the weakness of the gravitational interaction, they are not easily detected.

For the time being, we assume the existence of gravitational waves in general relativity and study their properties. We will learn how they arise later in the course.

Properties of Gravitational Waves

- They travel at the speed of light (c)
- They are transverse to the propagation direction
- They have two independent polarisation states ($+$ and $\times$)
- They carry energy away from a radiating system
- They can be detected by their effects on freely-falling test particles

A Linearized Gravitational Wave

Simplest example: a ripple in the curvature of space-time propagating in one direction, independent of the other two:

- direction of propagation (z) is longitudinal direction
- other two directions (x and y) are transverse
- it is a plane wave

We assume that the geometry is close to flat and treat the gravitational wave as a perturbation, so

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{2.1}$$

A Linearized Gravitational Wave

- A simple example of gravitational wave propagating along z is:

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z). \qquad (2.2)$$

  where we require $|f(t-z)| \ll 1$.

- Typical amplitudes $|f(t-z)| \lesssim 10^{-21}$. Linearized approximation is fine: we are ignoring terms of size 10^{-40}!

A Linearized Gravitational Wave

The line element for the spacetime is

$$ds^2 = -dt^2 + [1 + f(t-z)]\,dx^2 + [1 - f(t-z)]\,dy^2 + dz^2 \tag{2.3}$$

$f(t-z)$ encodes the size and shape of the gravitational wave. Note that f and $h_{\alpha\beta}$ are dimensionless.

Examples:

1. $f(t-z) = a\sin[\omega(t-z)]$ is a gravitational wave of amplitude a and frequency ω.
2. $f(t-z) = b\exp[-(t-z)^2/2\sigma^2]$ is a Gaussian wave packet with width σ and amplitude b.

Detecting Gravitational Waves

- Spacetime curvature is detectable through the motion of test bodies, i.e., ones that move along geodesics and produce no significant spacetime perturbation.
- Curvature detection requires monitoring the relative motion of at least two test bodies.
- Let's examine the motion of two bodies, initially at rest, as a gravitational wave passes. We need to examine both their co-ordinate motion and their proper distance.

Exercise 2

Consider two particles A and B at positions $(0, 0, 0)$ and (x_B, y_B, z_B) initially at rest.

(a) What are their initial 4-velocities?

(b) Use the geodesic equation to determine the evolution of the co-ordinate positions of the particles when a gravitational wave of the form

$$ds^2 = -dt^2 + [1 + f(t-z)]\, dx^2 + [1 - f(t-z)]\, dy^2 + dz^2$$

passes.

(c) Now, consider particle B at $(L_\star, 0, 0)$. What happens to the proper distance between the particles as the gravitational wave passes.

Distances Do Change

- The fractional change in distance δL is

$$\frac{\delta L(t)}{L_\star} := \frac{L(t) - L_\star}{L_\star} = \frac{1}{2} f(t) = \frac{1}{2} h_{xx}(t)$$

- For a gravitational wave of amplitude a and angular frequency ω, we obtain

$$\frac{\delta L(t)}{L_\star} = \frac{a}{2} sin(\omega t).$$

The fractional distance oscillates with half the amplitude and the same frequency of the gravitational wave.

Distances Do Change

- For a particle at $(0, L_\star, 0)$, we have

$$\frac{\delta L(t)}{L_\star} = -\frac{1}{2} f(t) = \frac{1}{2} h_{yy}(t)$$

- More generally, for a particle at distance $L_\star$ from the origin, in direction $\hat{n}^i$ in the z=0 plane,

$$\frac{\delta L(t)}{L_\star} = \frac{1}{2} h_{ij}(t) \hat{n}^i \hat{n}^j$$

- $\frac{\delta L(t)}{L_\star}$ is the fractional strain produced by the gravitational wave.

A Ring of Particles

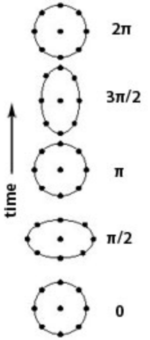
Now, consider a perturbation

$$ds^2 = -dt^2 + [1 + f(t-z)]\,dx^2 + [1 - f(t-z)]\,dy^2 + dz^2$$

and a ring of particles $\vec{n} = L_\star(\cos\theta, \sin\theta, 0)$. Then,

$$\frac{\delta L(t)}{L_\star} = \frac{1}{2} f(t)(\cos^2\theta - \sin^2\theta) = \frac{1}{2} f(t) \cos 2\theta$$

This is an ellipse with semi-major/minor axis $L_\star(1 \pm \frac{1}{2}|f(t)|)$.



A Second Polarisation

So far, we considered only a single gravitational wave polarisation:

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z) \quad , \quad |f(t-z)| \ll 1\,. \tag{2.4}$$

This is not the most general possible form of a gravitational wave travelling in the z-direction. This can by shown by considering a change of co-ordinates.

Exercise 3

Starting from the perturbed metric:

$$ds^2 = -dt^2 + [1 + f(t-z)]\, dx^2 + [1 - f(t-z)]\, dy^2 + dz^2$$

consider a change of variables to x', y' given by

$$x = \frac{1}{\sqrt{2}}(x' + y') \quad y = \frac{1}{\sqrt{2}}(y' - x')$$

and derive the metric in these co-ordinates and the form of $h_{\alpha\beta}$.

Two Polarisations

The general solution for a GW propagating in the z direction is

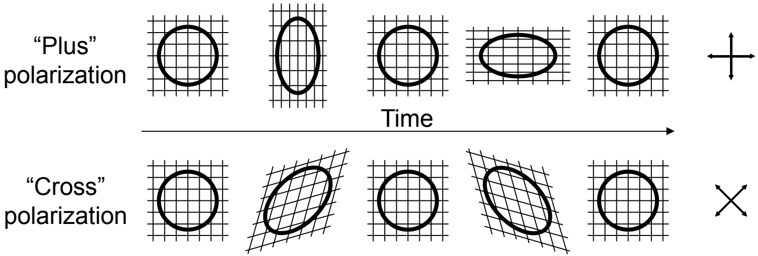
$$h_{\alpha\beta}(t,z) = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} h_{+}}_{\substack{\textbf{“plus”} \\ \textbf{polarisation}}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} h_{\times}}_{\substack{\textbf{“cross”} \\ \textbf{polarisation}}} . \qquad (2.5)$$

where $h_{+}(t-z)$ and $h_{\times}(t-z)$ are arbitrary functions.

[We have not proved this, but it is not difficult.]

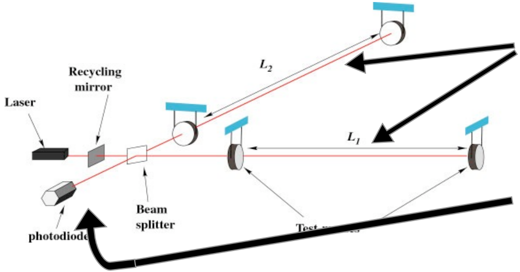
The Effect of Gravitational Waves

Gravitational waves are **deformations of space** itself that stretch it first in one direction, then in the perpendicular one.



Detecting Gravitational Waves: Interferometers

A laser interferometer is shaped like an L to take advantage of the quadrupolar deformations produced by a passing gravitational wave.



As a wave passes, the arm lengths change in different ways …

… causing the interference patter at the photodiode to change.

Interferometer Basics

- We have seen that a plus polarised wave propagating in the z-direction will cause an alternate expansion and contraction of distances in the x and y directions.
- A difference in length of the two arms will lead to an interference pattern:
    1. Constructive interference when
    $$\Delta L = L_{(x)} - L_{(y)} = n\lambda, \quad n = 0, 1, 2, \ldots$$
    2. Destructive interference when
    $$\Delta L = \left(n + \frac{1}{2}\right)\lambda, \quad n = 0, 1, 2, \ldots$$
- A path length difference will manifest as an interference pattern at the output photodiode.

Effect of Gravitational Waves

- A passing gravitational wave will change the lengths of the arms and cause an interference pattern to be formed at the detector output.
- Taking a + polarised gravitational wave propagating in the z-direction,

$$\frac{\delta L_x}{L_x} = +\frac{1}{2} a \sin(\omega t) \quad \frac{\delta L_y}{L_y} = -\frac{1}{2} a \sin(\omega t) \tag{2.6}$$

- The amplitude and frequency can be inferred from the observed interference pattern.
- Since the change in length is proportional to the length, a longer interferometer will be more sensitive

Gravitational Wave Detectors

The above is a simplified discussion of gravitational wave detectors. More details will follow later in the course. In particular:

- When the arm lengths are equal there is actually destructive interference. This is due to phase changes of the light when it is reflected by the mirrors.
- We have not discussed the additional mirrors used to build up power in the arms and increase detector sensitivity.
- We have not discussed noise sources and detector sensitivity.

Gravitational Wave Detectors

Several kilometer-scale interferometers have operated since 2005, alternating periods of observing and upgrades:

- Two LIGO detectors in the US: Livingston, LA & Hanford, WA (4 km arms). Made the 1^{st} detection in Sept. 2015.
- The Virgo detector in Pisa, Italy (with 3 km arms). Observed with LIGO during August 2017 and participated in two detections.
- The GEO-600 detector in Hannover, Germany (with 600 m arms). Primarily a technology development platform, also participated in observing up to 2010.

LIGO and Virgo have a sensitivity about

$$\delta L/L < 10^{-22}$$
$$\delta L < 10^{-19}\mathrm{m} \sim 1/1000 \text{ the diameter of a proton.}$$

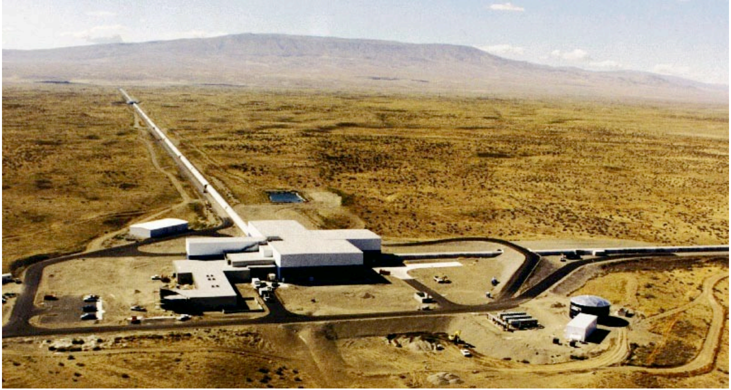
Gravitational Wave Detectors

The detections made so far are all mergers of black holes (BBH) or neutron stars (BNS). They include:

- GW150914: BBH, total mass 65 $M_\odot$. HL
- GW151226: BBH, total mass 22 $M_\odot$. HL
- GW170104: BBH, total mass 51 $M_\odot$. HL
- GW170608: BBH, total mass 19 $M_\odot$. HL
- GW170814: BBH, total mass 55 $M_\odot$. HLV
- GW170817: BNS, EM counterpart observed in gamma, X, optical, IR, radio bands. HLV

KAGRA a new underground detector in Japan will be online in 2019-2020. LIGO-India is currently selecting its site, might be observing by 2025.

LIGO Hanford Observatory



Energy in General Relativity

- The energy density in Newtonian gravity is

$$\epsilon_{\mathrm{Newt}} = -\frac{1}{8\pi G}[\nabla\Phi(x)]^2 \tag{2.7}$$

- One might try to generalise the above by replacing $\nabla\Phi(x)$ by derivatives of the metric. But, physical quantities in GR cannot depend upon the choice of co-ordinates, and we can choose co-ordinates where they vanish.
- There is no local notion of energy in a gravitational field in general relativity (in contrast to Newtonian gravity).
- This absence is part of the profound shift in viewpoint from gravity as a force field operating **in** spacetime to gravity **as** spacetime.

Energy in Gravitational Waves

- We can associate energy to an entire spacetime when it is asymptotically flat, e.g. black hole masses.
- We can obtain an approximate expression, when the wavelength of the waves is much smaller than the scale of curvature. This energy is an average energy density over spacetime volumes the dimensions of which are larger than the wavelength but much smaller than the background curvature scale.

Energy in Gravitational Waves

We can get an answer based on simple arguments. Consider the waveform $f(t-z) = a\sin(\omega(t-z))$. Then:

- Assume that energy density is proportional to the amplitude squared, $\epsilon \propto a^2$ (as for other waves).
- The units of energy density are $ML^{-1}T^{-2}$.

Exercise 4

(a) The units of energy density are $ML^{-1}T^{-2}$.

(b) What (unique) combination of ω, G and c has the same dimensions as energy density?

Energy in Gravitational Waves

We can get an answer based on simple arguments. Consider the waveform $f(t-z) = a\sin(\omega(t-z))$. Then:

- Assume that energy density is proportional to the amplitude squared, $\epsilon \propto a^2$ (as for other waves).
- The units of energy density is $ML^{-1}T^{-2}$. The only way to obtain these dimensions is with $\epsilon \propto \frac{c^2\omega^2}{G}$.
- The energy density of a gravitational wave with amplitude a and angular frequency ω is

$$\epsilon_{\rm GW} = \frac{c^2\omega^2 a^2}{32\pi G}$$

- Their speed is c, so their energy flux is

$$f_{\rm GW} = \epsilon_{\rm GW} c = \frac{c^3\omega^2 a^2}{32\pi G}$$

Exercise 5

(a) For a typical gravitational wave that advanced LIGO aims to detect, $a \approx 10^{-22}$ and the frequency $f \approx 200\mathrm{Hz}$. Substitute these into the expression:

$$f_{GW} = \frac{c^3 \omega^2 a^2}{32\pi G}$$

to calculate the typical energy flux of a gravitational wave.

(b) What would the luminosity of the source (assuming uniform emission) be if it were at:

- i The galactic centre ($\approx 10kpc$)
- ii The Virgo cluster ($\approx 10Mpc$)
- iii The Coma supercluster ($\approx 100Mpc$)

Lectures 5 - 7: “A Little More Math” [Hartle, Ch. 20]

Towards the Einstein Equation

The Einstein Equation is a set of differential equations for the metric; Schwarzschild metric, Kerr metric, gravitational waves, etc. are possible solutions.

Studying this equation requires three final mathematical concepts:

- A more rigorous definition of vectors in terms of **directional derivatives**.
- **Tensors**: generalizations of vectors. Physically measurable quantities in GR are tensors (e.g., metric, curvature).
- **Covariant derivative**: how we take derivatives of tensors (for making differential equations).

Vectors

- We defined a vector as a tangent to a curve $x^\alpha = x^\alpha(\sigma)$ at a point (event) P. The vector $\mathbf{t}$ has components

$$t^\alpha := \left.\frac{dx^\alpha}{d\sigma}\right|_P \tag{3.1}$$

  with respect to a coordinate basis $\{\mathbf{e}_\alpha\}$ associated with x^α.

- A vector defined this way is sometimes called a **tangent vector**.

Vectors as directional derivatives

We can also consider **t** as a **directional derivative** operator.

- Consider a function $f = f(x^\alpha)$ and a curve $x^\alpha(\sigma)$.
- The directional derivative along the curve, at the point labelled by σ is

$$\frac{df}{d\sigma} = \lim_{\epsilon \to 0} \left[\frac{f(x^\alpha(\sigma + \epsilon)) - f(x^\alpha(\sigma))}{\epsilon} \right] = \frac{dx^\alpha}{d\sigma} \frac{\partial f}{\partial x^\alpha} \quad (3.2)$$

- The vector **t** with coordinate basis components

$$t^\alpha = \frac{dx^\alpha}{d\sigma} \quad (3.3)$$

is a tangent vector to the curve.

Directional Derivatives

- The directional derivative at σ is specified by $\mathbf{t}$ and we can write
$$\frac{d}{d\sigma} = t^\alpha \frac{\partial}{\partial x^\alpha} \,. \tag{3.4}$$
- There is a one to one correspondence between vectors and directional derivatives:
$$\mathbf{a} = a^\alpha \frac{\partial}{\partial x^\alpha} \tag{3.5}$$

Directional Derivatives

- We could have used this definition in flat space. Given a directional derivative, we can use it to construct a directed line segment. But, we do not need to!
- On the other hand, directed line segments do not work in curved space. Directional derivatives do.
- We can just work with directional derivatives. Usual vector algebra works. Simply think of

$$\mathbf{e}_\alpha = \frac{\partial}{\partial x^\alpha} \tag{3.6}$$

  as a coordinate basis.
- From now on, think of vectors as directional derivatives. The linear space of directional derivatives is the tangent space mentioned before.

Example: Change of Co-ordinates

How do the components of a vector **a** change under a co-ordinate transformation $x^\alpha \to x'^\alpha(x^\beta)$?

- We can calculate

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta}$$

- Use this to calculate the components of **a**

$$\mathbf{a} = a^\alpha \frac{\partial}{\partial x^\alpha} = a^\alpha \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta} = a'^\beta \frac{\partial}{\partial x'^\beta}$$

- So we obtain

$$a'^\beta = \left(\frac{\partial x'^\beta}{\partial x^\alpha}\right) a^\alpha \quad \text{and} \quad a^\alpha = \left(\frac{\partial x^\alpha}{\partial x'^\beta}\right) a'^\beta \qquad (3.7)$$

Example: Cartesian and Plane Polar Coordinates

- Consider 2-d flat space in either Cartesian $x^\alpha = (x, y)$ or polar coordinates $x'^\alpha = (r, \phi)$.
- Transformation equations $x'^\alpha = x'^\alpha(x^\alpha)$:

$$r = \sqrt{x^2 + y^2}\,, \quad \phi = \arctan(y/x)$$

- Inverse transformation equations $x^\alpha = x^\alpha(x'^\alpha)$:

$$x = r\cos\phi\,, \quad y = r\sin\phi$$

Example: Cartesian and Plane Polar Coordinates

- Transformation matrix:

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix}$$

- Inverse transformation matrix:

$$\frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}$$

- We can express the polar coordinate basis vectors as

$$\begin{cases} \mathbf{e}_r &= \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y \\ \mathbf{e}_\phi &= -r\sin\phi\,\mathbf{e}_x + r\cos\phi\,\mathbf{e}_y \end{cases}$$

Dual Vectors

- **Dual vectors** (or co-vectors or one-forms) are defined as linear mappings from the space of vectors V at a point P to the real numbers — i.e., $\boldsymbol{\omega} : V \rightarrow \mathbb{R}$ satisfying:
  1. $\omega(\mathbf{a}+\mathbf{b}) = \omega(\mathbf{a}) + \omega(\mathbf{b})$, $\forall$ $\mathbf{a}, \mathbf{b}$.
  2. $\omega(\alpha\mathbf{a}) = \alpha\omega(\mathbf{a})$, $\forall$ $\alpha \in \mathbb{R}$ and $\forall$ $\mathbf{a}$.

- Using linearity, and assuming zero maps to zero, one can write

$$\omega(\mathbf{a}) = \omega(a^\alpha \mathbf{e}_\alpha) = a^\alpha \omega(\mathbf{e}_\alpha) = a^\alpha \omega_\alpha \tag{3.8}$$

where $\omega_\alpha := \omega(\mathbf{e}_\alpha)$ are defined to be the components of the dual vector $\boldsymbol{\omega}$ with respect to the x^α coordinate basis.

The Gradient as a Dual Vector

- The derivative of a function in the direction specified by a vector $\mathbf{t}$ is
$$\frac{df}{d\sigma} = t^{\alpha}\frac{\partial f}{\partial x^{\alpha}} \tag{3.9}$$

- The derivatives of $f(x^{\alpha})$ specify a map from any vector $\mathbf{t}$ to real numbers:
$$\nabla_{\mathbf{t}} f = \mathrm{d}f\left(\frac{d}{d\sigma}\right) = t^{\alpha}\frac{\partial f}{\partial x^{\alpha}} \tag{3.10}$$

- Therefore, the gradient of a function, $\boldsymbol{\nabla} f$, is a dual vector with components
$$(\nabla f)_{\alpha} = \frac{\partial f}{\partial x^{\alpha}} \tag{3.11}$$

Basis of Dual Vectors

- We can introduce a set of linearly independent dual vectors $\mathbf{e}^\alpha$, and write any dual vector as cotangent space

$$\boldsymbol{\omega} = \omega_\alpha \mathbf{e}^\alpha$$

- The numbers ω_α are the components of the dual vector in the basis $\mathbf{e}^\alpha$.
- It is most convenient to work with a dual basis that satisfies

$$e^\alpha(\mathbf{e}_\beta) = \delta^\alpha_\beta \tag{3.12}$$

i.e., the coordinate basis vectors and dual basis vectors are dual to one another.

Example: Change of Co-ordinates

How do the components of a dual vector $\boldsymbol{\omega}$ change under a co-ordinate transformation $x^\alpha \to x'^\alpha(x^\beta)$?

- We know the $\boldsymbol{\omega}(\mathbf{a})$ is independent of the choice of co-ordinates so, using (3.7)

$$\begin{aligned}\boldsymbol{\omega}(\mathbf{a}) = \omega_\alpha a^\alpha &= \omega'_\beta a'^\beta \\ &= \omega'_\beta \left(\frac{\partial x'^\beta}{\partial x^\alpha}\right) a^\alpha\end{aligned}$$

- Therefore, the components of a dual vector transform as

$$\omega'_\alpha = \left(\frac{\partial x^\beta}{\partial x'^\alpha}\right)\omega_\beta \quad \text{and} \quad \omega_\alpha = \left(\frac{\partial x'^\beta}{\partial x^\alpha}\right)\omega'_\beta \qquad (3.13)$$

Identifying Vectors and Dual Vectors

- The metric $g_{\alpha\beta}$ allows one to identify vectors with dual vectors and vice-versa.
- For a fixed vector **a** the inner product defines a map from vectors **b** to real numbers as

$$a(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta \tag{3.14}$$

- Hence,

$$a_\beta := g_{\alpha\beta} a^\alpha \tag{3.15}$$

  are the components of a dual vector associated with **a**.
- This gives a correspondence between the vector with components a^α and the dual vector with components a_α.

Identifying Vectors and Dual Vectors

- To transform a dual vector to a vector, we require the inverse of the metric, defined by

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}_{\beta} \tag{3.16}$$

- Then, given the components ω_β of a dual vector, one can associate a vector with components

$$\omega^{\alpha} := g^{\alpha\beta} \omega_{\beta} \tag{3.17}$$

- The correspondence between a vector and dual vector via $a_\alpha := g_{\alpha\beta} a^\beta$ is called **lowering of an index with the metric**.
- The correspondence between a dual vector and vector via $a^\alpha := g^{\alpha\beta} a_\beta$ is called **raising of an index with the metric**.

Exercise 6

Consider the two dimensional metric

$$g_{\alpha\beta} = \begin{pmatrix} F & 1 \\ 1 & 0 \end{pmatrix}$$

and the vectors (or dual vectors)

$$a_\alpha = (1,0) \quad b_\alpha = (0,1) \quad c^\alpha = (1,0) \quad d^\alpha = (0,1)$$

Calculate

1. $g^{\alpha\beta}$
2. $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{c} \cdot \mathbf{c}$
3. $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{c} \cdot \mathbf{d}$

Example: Normal and Tangent Vectors

An example of the correspondence is the association of a **normal vector n** with the (dual vector) gradient of a function $f(x^\alpha) = \mathrm{const}$ defining a 3-d surface in spacetime—i.e.,

$$n^\alpha := g^{\alpha\beta} \frac{\partial f}{\partial x^\beta}$$

Note that for any vector **t** tangent to the surface

$$\mathbf{t} \cdot \mathbf{n} = g_{\alpha\beta} t^\alpha n^\beta = g_{\alpha\beta} t^\alpha g^{\beta\mu} \frac{\partial f}{\partial x^\mu} = t^\alpha \frac{\partial f}{\partial x^\alpha} = 0$$

Tensors

- Tensors are a **generalization of vectors & dual vectors**.
- We defined a dual vector as a linear mapping from the space of vectors V at point P to $\mathbb{R}$.
- We could equally consider a linear map from a pair of vectors at P to $\mathbb{R}$. We already met an of this:

$$g(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta} a^{\alpha} b^{\beta} \tag{3.18}$$

- The general notion is called a **tensor**.

Tensors

A **tensor of rank r** is a linear map from r vectors to $\mathbb{R}$.

- A scalar is a tensor of rank 0.
- A dual vector is a tensor of rank 1.
- The metric is a tensor of rank 2.
- A third rank tensor **t** is a linear map from 3 vectors to $\mathbb{R}$, for example

$$t(\mathbf{a}, \mathbf{b}, \mathbf{c}) = t_{\alpha\beta}{}^{\gamma} a^{\alpha} b^{\beta} c_{\gamma} \tag{3.19}$$

We might consider maps from n vectors and m dual vectors to $\mathbb{R}$, but since we have learnt how to associate vectors and dual vectors, this is not needed.

Operations on Tensors (and Examples)

- **Raising and lowering of indices with the metric:**
  1. $a^\alpha := g^{\alpha\beta} a_\beta$ converts a dual vector to a vector.
  2. We can raise both indices on a rank 2 tensor: $t^{\alpha\beta} := g^{\alpha\gamma} g^{\beta\delta} t^\gamma{}_{\gamma\delta}$

- **Summing over a pair of indices (or "contracting"):** lowers the rank of a tensor by two: $t^\mu{}_{\alpha\mu\beta}$ is a rank 2 tensor made from a rank 4 tensor $t^\mu{}_{\alpha\gamma\beta}$.

- **Product of two tensors:** $t^{\alpha\beta} = a^\alpha b^\beta$ are the components of a rank 2 tensor constructed from two rank 1 tensors **a** and **b**.

Coordinate Transformations

The action of tensors on vectors is invariant under coordinate transformations, we can work out how the components of a tensor change (just like we did for dual vectors).

Example: metric

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \tag{3.20}$$

General expression

$$T'^{\alpha\beta\ldots}_{\gamma\delta\ldots} = T^{\mu\nu\ldots}_{\lambda\sigma\ldots} \left(\frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \ldots \right) \left(\frac{\partial x^\lambda}{\partial x'^\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \ldots \right) . \tag{3.21}$$

Exercise 7

Starting from the 2-d cartesian metric,

$$ds^2 = dx^2 + dy^2 \, ,$$

make use of the transformation rules for the metric $g_{\alpha\beta}$, to calculate the metric in polar coordinates (r, ϕ).

Coordinate Transformations

- If the components of a tensor vanish with respect to one basis, they vanish with respect to **any other basis**.
- If you can show the equality of the components of two tensors in one basis, then the two tensors are equal. Choose coordinate systems to make your life easy!
- Not everything we have worked with is a tensor! Most importantly, coordinates x^α and the Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$ are not tensors.

So What is the Point of Tensors?

- **A physical law written as a tensor equation is valid in all coordinate systems** – it is invariant. E.g.: The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \,. \tag{3.22}$$

  It takes this same form in any choice of coordinates.

- If we write laws using tensors, they will automatically obey the principle of relativity in **all** frames (not just locally inertial ones).

Covariant Differentiation

- We have seen earlier that the partial derivative of a function f is a dual vector $\boldsymbol{\nabla} f$ with components

$$\nabla_\alpha f = \frac{\partial f}{\partial x^\alpha} \tag{3.23}$$

- It would be useful to have a similar definition for a vector, i.e. we would like to be able to take the derivative of a vector $\boldsymbol{\nabla} \boldsymbol{v}$ and get a rank 2 tensor.
- Similarly for tensors of other ranks.
- This would give us a definition of differentiation which is co-ordinate independent.

Parallel Transport

- To compute the derivative of a vector, we need to subtract the vector evaluated at nearby space-time points.
- But, vectors at different points live in different tangent spaces $\Rightarrow$ we cannot subtract them directly.
- We must find a way to transport vectors from one point to another.
- In flat space, we would simply do this by keeping the co-ordinate components constant as we move the vector. This is called parallel transport.
- We can generalise this to curved space by working in a local inertial frame.

Parallel Transport and Covariant Differentiation

Covariant derivative of a vector field

$$\boldsymbol{\nabla}_{\boldsymbol{t}} \boldsymbol{v}(x^\alpha) = \lim_{\epsilon \to 0} \frac{\mathbf{v}(x^\alpha + \epsilon t^\alpha)_{||\ \text{transport to}\ x^\alpha} - \mathbf{v}(x^\alpha)}{\epsilon} \qquad (3.24)$$

The way to evaluate this, work in a local inertial frame:

$$(\boldsymbol{\nabla}_{\boldsymbol{t}} \boldsymbol{v})^\alpha = t^\beta \frac{\partial v^\alpha}{\partial x^\beta} \quad \text{or} \quad \nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} \quad \text{(LIF)} \qquad (3.25)$$

Covariant Differentiation

- In an arbitrary coordinate system, we have to take into account the change in the basis vectors as we move from point to point.
- To see why, note that $\frac{\partial v^\alpha}{\partial x^\beta}$ of the **non-constant** radial vector field $v^r(r,\phi) = 1$, $v^\phi(r,\phi) = 0$ is zero!
- Although the components of $\mathbf{v}$ are constant, the basis vectors in polar coordinates **change** from point to point.
- When we parallel transport a vector field, the components of the vector field can change proportionally to
  1. the components of the vector v^α
  2. the displacement dx^α

$$\Rightarrow \quad v^\alpha_{||}(x^\delta) = v^\alpha(x^\delta + dx^\delta) + \tilde{\Gamma}^\alpha{}_{\beta\gamma}(x^\delta) v^\gamma(x^\delta) dx^\beta(x^\delta)$$

Covariant Differentiation

- We now want to determine the coefficients $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$
- We can do this by considering geodesics. The unit tangent to a curve **u** must be unchanged under parallel propagation, i.e.

$$(\nabla_{\boldsymbol{u}}\boldsymbol{u})^{\alpha} = u^{\beta}\left(\frac{\partial u^{\alpha}}{\partial x^{\beta}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}u^{\gamma}\right) = 0 \qquad (3.26)$$

- But, we already know that **u** satisfies the geodesic equation.
- From this, it follows that $\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}$.

Covariant Differentiation

Covariant derivative

$$\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma \tag{3.27}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols.

Geodesic equation

We can also rewrite the geodesic equation as:
A geodesic is a curve which satisfies

$$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} = 0 \tag{3.28}$$

Covariant Derivative of Dual Vectors

We can easily derive this from what we know:

- The **covariant derivative of a scalar field** ϕ is just the ordinary partial derivative:

$$\nabla_\alpha \phi = \frac{\partial \phi}{\partial x^\alpha}$$

- The covariant derivative obeys the product rule, e.g.

$$\nabla_\alpha(\omega_\beta v^\beta) = (\nabla_\alpha \omega_\beta) v^\beta + \omega_\beta (\nabla_\alpha v^\beta) \qquad (3.29)$$

- This gives

$$\nabla_\alpha \omega_\beta = \frac{\partial \omega_\beta}{\partial x^\alpha} - \Gamma^\mu{}_{\beta\alpha} \omega_\mu \qquad (3.30)$$

Note the minus sign in front of the Christoffel symbol.

Covariant Derivative of Tensors

General case:

$$\begin{aligned} \nabla_\alpha \left(T^{\mu\nu\cdots}_{\lambda\sigma\cdots} \right) &= \frac{\partial}{\partial x^\alpha} \left(T^{\mu\nu\cdots}_{\lambda\sigma\cdots} \right) \\ &+ \left(\Gamma^\mu_{\alpha\beta} T^{\beta\nu\cdots}_{\lambda\sigma\cdots} + \Gamma^\nu_{\alpha\beta} T^{\mu\beta\cdots}_{\lambda\sigma\cdots} + \ldots \right) \\ &- \left(\Gamma^\beta_{\alpha\lambda} T^{\mu\nu\cdots}_{\beta\sigma\cdots} + \Gamma^\beta_{\alpha\sigma} T^{\mu\nu\cdots}_{\lambda\beta\cdots} + \ldots \right) . \quad (3.31) \end{aligned}$$

Covariant derivative = partial derivative + (1 Γ term for each raised index) − (1 Γ term for each lowered index).

Covariant Derivative of the Metric

- The covariant derivative of the metric is:

$$\nabla_\mu g_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial x^\mu} - \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} - \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} \tag{3.32}$$

- In a local inertial coordinate system, $\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = 0$, so $\Gamma^\mu{}_{\alpha\beta} = 0$ and

$$\nabla_\mu g_{\alpha\beta} = 0 \,. \tag{3.33}$$

- But since $\nabla_\mu g_{\alpha\beta} = 0$ is a valid tensor equation, it holds in all coordinate systems.

Exercise 8

Show that $\nabla_\mu g_{\alpha\beta} = 0$.
To do this, you will need to make use of the explicit form of the Christoffel symbols:

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right) \tag{3.34}$$

Lectures 8-10 : Curvature and the Einstein Equation [Hartle, Ch. 21]

The central result of general relativity: Einstein's Equation

$$\begin{pmatrix}\text{a measure of local} \\ \text{space-time curvature}\end{pmatrix} = \begin{pmatrix}\text{a measure of matter} \\ \text{energy density}\end{pmatrix} \tag{4.1}$$

- Colloquially: "Matter tells space how to curve, space tells matter how to move"
- In this section, we introduce curvature and the vacuum Einstein equation. Later, we introduce energy density.

Tidal Gravitational Forces

- A gravitational field can be detected by monitoring the separation of two freely-falling particles.
- In Newtonian gravity, the equation of motion for a single freely-falling particle moving in a gravitational potential $\Phi = \Phi(x^k)$ is

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\Phi}{\partial x^j} \tag{4.2}$$

- If a second freely-falling particle is displaced from the first by χ^i, where $|\chi^i| \ll 1$, then

$$\frac{d^2\chi^i}{dt^2} = -\delta^{ij}\left.\frac{\partial^2\Phi}{\partial x^j\partial x^k}\right|_x \chi^k \tag{4.3}$$

where the second partial derivative of Φ is evaluated at the location of the first particle.

Exercise 9

Show that for $\Phi = -GM/r$, if two freely-falling particles are separated by a small radial displacement χ, then

$$\frac{d^2\chi}{dt^2} = +\frac{2GM}{r^3}\chi$$

Similarly, if two freely-falling particles are separated by a small transverse displacement χ, then

$$\frac{d^2\chi}{dt^2} = -\frac{GM}{r^3}\chi$$

Hence a ring of freely-falling particles would be distorted into an ellipse (**stretched in the radial direction, compressed in the transverse direction**) as it fell toward the center of the gravitational potential.

Tidal Gravitational Forces

- The quantity

$$\frac{\partial^2 \Phi}{\partial x^j \partial x^k} \tag{4.4}$$

is called the **Newtonian tidal acceleration tensor**.

- Note that

$$\nabla^2 \Phi = \delta^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = 4\pi G \mu \tag{4.5}$$

is the field equation for Newtonian gravity.

Geodesic Deviation

In GR, the Riemann curvature tensor plays the role of the Newtonian tidal acceleration tensor.

To see this, let us consider two nearby geodesics.

- We take the proper time to be τ.
- Let $\mathbf{u}$ denote the tangent vector to the first geodesic.
- Let $\boldsymbol{\chi}$ denote the separation vector connecting points along the geodesics with equal values of τ.

Note: there is no unique way to specify the relative time on the two geodesics. E.g., we could require $\boldsymbol{\chi} \cdot \boldsymbol{u} = 0$ at $\tau = 0$.

Geodesic Deviation

In analogy with the Newtonian case, we need to calculate the second derivative, or "acceleration", of $\boldsymbol{\chi}$.

- The derivative of the vector $\boldsymbol{\chi}$ with respect to τ is given by the covariant derivative along **u**

$$\mathbf{v} := \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\chi} \tag{4.6}$$

- The second derivative of $\boldsymbol{\chi}$ with respect to τ is given by

$$\mathbf{w} := \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\chi} \tag{4.7}$$

We want an expression for $\mathbf{w}$ that is linear in χ and does not contain its derivatives. We achieve this by

1. Substituting
$$v^\alpha = \frac{d\chi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta \chi^\gamma$$
into
$$w^\alpha = \frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta v^\gamma$$
2. Eliminating the $\frac{d^2\chi}{d\tau^2}$ term by using the geodesic equation at $x^\alpha + \chi^\alpha$:
$$\frac{d^2(x^\alpha + \chi^\alpha)}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}(\mathbf{x} + \boldsymbol{\chi}) \frac{d(x^\beta + \chi^\beta)}{d\tau} \frac{d(x^\gamma + \chi^\gamma)}{d\tau} = 0$$
expanded to leading order in χ^α.
3. Using the geodesic equation at x^α to eliminate $\frac{d^2x}{d\tau^2}$.

Geodesic Deviation & Curvature

Hartle does the calculation in detail (posted in learning central) – recommend working through it. The result:

1. is linear in χ (and not its derivatives)
2. depends upon two factors of u^α (recall that $d/d\tau = u^\alpha \partial_\alpha$)
3. contains terms $\partial\Gamma$ and $\Gamma\Gamma$.

We obtain the geodesic deviation equation.

$$\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{u}} \chi^\alpha = -R^\alpha{}_{\beta\gamma\delta} u^\beta u^\delta \chi^\gamma \tag{4.8}$$

where $R^\alpha{}_{\beta\gamma\delta}$ are the components of the Riemann tensor.

Riemann Curvature Tensor

- The explicit expression for the Riemann curvature tensor is

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}{}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}{}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}{}_{\gamma\epsilon}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\epsilon}\Gamma^{\epsilon}{}_{\beta\gamma} \qquad (4.9)$$

- The Riemann tensor encodes the failure of initially parallel geodesics to remain parallel.

- In other words, tidal effects of gravitation can be attributed to the local curvature of spacetime itself, and not to some “mysterious” force called gravity.

Geodesic Deviation in a Freely Falling Frame

Consider a freely falling observer at $x^\alpha(\tau)$, moving on a geodesic.

- We can define an orthonormal basis $\{\mathbf{e}_{\hat{\alpha}}\}$ (where $\mathbf{e}_{\hat{\tau}} = \mathbf{u}$) at one point on the geodesic
- To extend to other points, we parallel propagate the basis vectors along the geodesic, i.e.

$$\nabla_{\boldsymbol{u}} \boldsymbol{e}_{\hat{\alpha}} = 0 \quad \text{and} \quad \nabla_{\boldsymbol{u}} \boldsymbol{e}^{\hat{\alpha}} = 0$$

- Then, we want to evaluate the geodesic deviation equation in these co-ordinates.

First, an aside on tensor components in an orthonormal frame. . .

Tensor Co-ordinates in an Orthonormal Basis

- Given a vector **a**, its components in a given basis are:

$$a^{\alpha} = \mathbf{e}^{\alpha} \cdot \mathbf{a} \quad \text{and} \quad a_{\alpha} = \mathbf{e}_{\alpha} \cdot \mathbf{a}\,,$$

  and similarly for tensors.
- For orthonormal bases:

$$a^{\hat{\alpha}} = \mathbf{e}^{\hat{\alpha}} \cdot a \quad \text{and} \quad a_{\hat{\alpha}} = \mathbf{e}_{\hat{\alpha}} \cdot a$$

  and for a rank 2 tensor,

$$t_{\hat{\alpha}\hat{\beta}} = (\mathbf{e}_{\hat{\alpha}})^{\alpha} (\mathbf{e}_{\hat{\beta}})^{\beta} t_{\alpha\beta}$$

  where $(\mathbf{e}_{\hat{\alpha}})^{\alpha}$ are the components of the orthonormal basis vectors in the coordinate system we are using.

Geodesic Deviation in a Freely Falling Frame

- We need to multiply both sides of the geodesic deviation equation

$$\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\chi^{\alpha} = -R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}u^{\delta}\chi^{\gamma}$$

by $(\mathbf{e}^{\hat{\alpha}})_{\alpha}$.

- By construction $\nabla_{\boldsymbol{u}}\mathbf{e}^{\hat{\alpha}} = 0$ so we can move it inside the derivatives.
- Also, $\mathbf{e}_{\hat{\tau}} = \mathbf{u}$, so we can write

$$\frac{d^2\chi^{\hat{\alpha}}}{d\tau^2} = -R^{\hat{\alpha}}{}_{\hat{\tau}\hat{\beta}\hat{\tau}}\chi^{\hat{\beta}}$$

where

$$R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} = R^{\alpha}{}_{\beta\gamma\delta}(\mathbf{e}^{\hat{\alpha}})_{\alpha}(\mathbf{e}_{\hat{\beta}})^{\beta}(\mathbf{e}_{\hat{\gamma}})^{\gamma}(\mathbf{e}_{\hat{\delta}})^{\delta}\,.$$

- This is remarkably similar to the Newtonian expression.

Exercise 10

For a weak static gravitational field described by the line element

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\left[dx^2 + dy^2 + dz^2\right]$$

show that

$$R_{jtkt} = \frac{\partial^2 \Phi}{\partial x^j \partial x^k}$$

to first-order in Φ.

Thus, in the weak field limit, we recover Newtonian gravity.

Hint: When evaluating the Riemann curvature, you can ignore the $\Gamma\Gamma$ terms as they will be quadratic in Φ.

Properties of the Curvature Tensor

- In a local inertial frame ($\Gamma = 0$, $\partial\Gamma \neq 0$), we can write the Riemann curvature as (∂_α is short-hand for $\partial/\partial x^\alpha$):

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\gamma\partial_\beta g_{\alpha\delta} - \partial_\gamma\partial_\alpha g_{\beta\delta} + \partial_\delta\partial_\alpha g_{\beta\gamma} - \partial_\delta\partial_\beta g_{\alpha\gamma}\right) \tag{4.10}$$

- This reveals the symmetries of the Riemann tensor:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} \\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma} \\ R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} \\ R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} &= 0 \end{aligned}$$

- Rather than 4^4 independent components in 4-dimensions, there are only 20. In 3-d there are 6 and in 2-d only 1.

Einstein Equation

- The vacuum field equation for Newtonian gravity is

$$\nabla^2\Phi \equiv \delta^{ij}\frac{\partial^2\Phi}{\partial x^i \partial x^j} = 0\,. \qquad (4.11)$$

- Einstein proposed a similar equation for General Relativity:

Vacuum Einstein equation

$$g^{\mu\nu}R_{\nu\alpha\mu\beta} = R^{\mu}{}_{\alpha\mu\beta} = 0\,. \qquad (4.12)$$

- The Ricci tensor is defined by

$$R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta} \qquad (4.13)$$

The vacuum Einstein equation can be written as $R_{\alpha\beta} = 0$.

Einstein Equation

1. The Einstein equation (4.12) is a set of 10 **coupled**, **non-linear**, **second-order**, **partial differential equations** for the metric components $g_{\alpha\beta}$.

2. There is no systematic way to solve a system of coupled, non-linear partial differential equations. **Very few analytic solutions exist.** These correspond to situations with a high degree of symmetry.

3. More complicated situations require numerical solution of the field equations. There is no analytic solution to the two-body problem in GR.

Examples of Solutions to the Einstein Equation

1. The Schwarzschild solution exterior to a spherically-symmetric star or black hole.
2. The Kerr solution exterior to a black hole (i.e., an axially-symmetric spacetime). Does not hold for stars.
3. Gravitational waves (a weak gravitational field far from a non-stationary relativistic source).
4. Friedmann-Robertson-Walker cosmological solution for an homogeneous and isotropic universe, etc.

Linearized Gravity

We want to show that the linearized gravity metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \,, \tag{4.14}$$

where $|h_{\alpha\beta}| \ll 1$,

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} h_+(t-z) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} h_\times(t-z)$$

and $h_+(t-z)$ and $h_\times(t-z)$ are arbitrary functions, is a solution to the (linearized) Einstein equations.

Linearized Einstein Equations

- It is easy to show that the Riemann and Ricci curvature tensors vanish for flat space-time, $\eta_{\alpha\beta}$. So, flat space satisfies the Einstein equation.
- We will expand everything to leading order in $h_{\alpha\beta}$, ignoring anything of order h^2.
- We need to calculate the linearized Riemann and Ricci tensors and show that they satisfy the Einstein equations.
- Note: for quantities that vanish in the background, we denote the leading order terms with a δ, for example $\delta R^{\alpha}{}_{\beta\gamma\delta}$.

Exercise 11

Show that to first order in $h_{\alpha\beta}$ the Riemann tensor has components

$$\delta R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\gamma \partial_\beta h_{\alpha\delta} - \partial_\gamma \partial_\alpha h_{\beta\delta} + \partial_\delta \partial_\alpha h_{\beta\gamma} - \partial_\delta \partial_\beta h_{\alpha\gamma}\right)$$

where ∂_α is short-hand for $\partial/\partial x^\alpha$.

Linearized Einstein Equations

The linearized Ricci tensor is:

$$\begin{aligned}\delta R_{\alpha\beta} &= \delta R^{\mu}{}_{\alpha\mu\beta} \\ &= \frac{1}{2}\left(\partial_\mu\partial_\alpha h^\mu_\beta - \partial^\mu\partial_\mu h_{\alpha\beta} + \partial_\mu\partial_\beta h^\mu_\alpha - \partial_\alpha\partial_\beta h^\mu_\mu\right) \\ &= \frac{1}{2}\left(-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\right) \qquad (4.15)\end{aligned}$$

where

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2 \qquad (4.16)$$

is the so-called **D'Alembertian** (or wave operator) and

$$V_\alpha := \partial_\beta h^\beta_\alpha - \frac{1}{2}\partial_\alpha h^\beta_\beta \qquad (4.17)$$

Raising and Lowering Indices

- Note that for weak gravitational fields, one typically raises and lowers indices with the background Minkowski metric $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, and not with $g^{\alpha\beta}$ and $g_{\alpha\beta}$. For example,

$$h^{\alpha}{}_{\beta} := \eta^{\alpha\mu} h_{\mu\beta} \,, \quad h^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu} \,. \tag{4.18}$$

- The only exception is $g^{\alpha\beta}$, which still denotes the inverse of $g_{\alpha\beta}$, not $\eta^{\alpha\mu}\eta^{\beta\nu}g_{\mu\nu}$. To first order

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \tag{4.19}$$

A Word About Coordinate Transformations

- It is always possible to find coordinates for which the above decomposition is not valid — e.g., flat spacetime in spherical polar coordinates does not satisfy (4.14), even though the gravitational field is identically zero!
- The set of coordinates x^α in which (4.14) holds is **not** unique. It is possible to make an **infinitesimal coordinate transformation** $x^\alpha \to x'^\alpha$ for which the decomposition with respect to the new set of coordinates still holds.
- We will use this gauge freedom to our advantage to simplify calculations.

Choice of Coordinates

Consider an **infinitesimal coordinate transformation**

$$x'^\alpha := x^\alpha + \xi^\alpha(x) \quad \text{where} \quad |\partial_\alpha \xi^\beta| \ll 1 \tag{4.20}$$

- To first-order, the transformation matrix from x'^α to x^β is

$$\frac{\partial x^\beta}{\partial x'^\alpha} = \delta^\beta{}_\alpha - \frac{\partial \xi^\beta}{\partial x'^\alpha} = \delta^\beta{}_\alpha - \frac{\partial \xi^\beta}{\partial x^\alpha} \tag{4.21}$$

- To first order, the metric components transform as

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} = g_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$$

or, equivalently,

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha \tag{4.22}$$

Since $|\partial_\alpha \xi_\beta| \ll 1$, it follows that $|h'_{\alpha\beta}| \ll 1$.

Choice of Coordinates – Lorentz Gauge

- It is **always** possible to find a set of coordinates for which

$$V_\alpha = \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta = 0$$

  This is sometimes called the **Lorentz condition** (in analogy with the Lorentz condition in electromagnetism).

- To see this, consider how V_α transforms under an infinitesimal coordinate transformation:

$$V'_\alpha = V_\alpha - \partial_\beta \partial^\beta \xi_\alpha \, .$$

  So, to set $V'_\alpha = 0$, we need to solve the wave equation for ξ_α:

$$\Box \xi_\alpha = V_\alpha$$

Linearized Gravity

- In the Lorentz gauge, the equations for linearized gravity are

$$\Box h_{\alpha\beta} = 0 \tag{4.23}$$

  with the additional condition

$$V_\alpha = \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta = 0 \tag{4.24}$$

- Thus, the metric perturbations satisfy the flat space wave equation; hence the solutions can be interpreted as **gravitational waves**.

Solving the Wave Equation

- First, let's recall the wave equation for scalar functions

$$\Box f = 0 \,. \tag{4.25}$$

- This has the solution

$$f(x) = a e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{where} \quad \mathbf{k}\cdot\mathbf{k} = 0 \,. \tag{4.26}$$

- The most general solution is a superposition of plane waves.

Gravitational Wave Solution

- The most general gravitational wave solution is a linear combination of plane waves:

$$h_{\alpha\beta} = a_{\alpha\beta}\,\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{4.27}$$

where k^α satisfies:

$$\eta_{\alpha\beta}k^\alpha k^\beta = 0 \tag{4.28}$$

i.e., gravitational waves propagate at the speed of light.

- $h_{\alpha\beta}$ must also satisfy the Lorentz condition:

$$V_\alpha = \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta = 0$$

Transverse Traceless Gauge

- We have additional coordinate freedom to simplify things further: any coordinate transformation satisfying

$$\Box\xi_\alpha = 0 \Leftrightarrow \xi_\alpha = B_\alpha e^{i\mathbf{k}\cdot\mathbf{x}}$$

  will preserve the Lorentz condition.
- Under such a transformation, $a'_{\alpha\beta} = a_{\alpha\beta} - ik_\alpha B_\beta - ik_\beta B_\alpha$ and we can choose the B_α so that

$$a'_{ti} = 0 \quad a'^{\beta}_{\beta} = 0 \tag{4.29}$$

- The second of these sets the trace of $a_{\alpha\beta}$ to zero.

Transverse Traceless Gauge

- With $a_{ti} = 0$ and $a^{\beta}_{\beta} = 0$, the Lorentz condition simplifies:

$$a_{tt} = 0 \quad \text{and} \quad k^i a_{ij} = 0 \tag{4.30}$$

- The second of these ensures the wave is transverse: there is no perturbation in the direction of propagation.
- These 8 conditions define the transverse traceless (TT) gauge.
- We can simplify things further by choosing the z-direction as the direction of propagation of the wave, so that $k^{\alpha} = (k, 0, 0, k)$, in which case we have:

$$a_{tt} = 0 \qquad a_{ti} = 0$$
$$a_{iz} = 0 \qquad a_{xx} + a_{yy} = 0$$

Transverse Traceless gauge

- The metric perturbations $h_{\alpha\beta}$ in the TT gauge are thus

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)} \qquad (4.31)$$

- This is the most general solution of the linearized Einstein equation with definite frequency ω.

- The most general solution is a superposition of waves with different directions of propagation, amplitudes and frequencies.

Lectures 11-12: The Source of Curvature [Hartle, Ch. 22]

In the previous section, we introduced

Einstein's Equation

$$\begin{pmatrix}\text{a measure of local} \\ \text{space-time curvature}\end{pmatrix} = \begin{pmatrix}\text{a measure of matter} \\ \text{energy density}\end{pmatrix}$$

And discussed in detail the left hand side (curvature). Here, we move on to the right hand side (matter and energy density).

- We want to derive an expression for the stress-energy density in the Einstein equation.
- Start with an easier concept: number density

Number density

- Consider a box of volume $\mathcal{V}_\star$ at rest containing $\mathcal{N}$ particles. The number density is

$$n = \mathcal{N}/\mathcal{V}_\star \tag{5.1}$$

- Now, consider an observer moving with velocity V relative to the box. The length of the box in that direction will be contracted by a factor $(1 - V^2)^{1/2}$ so the volume is

$$\mathcal{V} = \mathcal{V}_\star(1 - V^2)^{1/2} \tag{5.2}$$

  and the number density is

$$N = \mathcal{N}/\mathcal{V} = n(1 - V^2)^{-1/2} \tag{5.3}$$

Number density

The **number density** can be written as $N = nu^t$, where u^t is the time component of the 4-velocity of the box. This motivates introducing the

Number current 4-vector

$$\mathbf{N} = n\mathbf{u} \tag{5.4}$$

The spatial parts describe the **number current density**:

$$\vec{N} = n\vec{u} = \frac{n\vec{V}}{\sqrt{1 - V^2}} \tag{5.5}$$

describing the flow of particles across a surface (per unit area per unit time).

Conservation of Particles

By shrinking the volume $\mathcal{V}_\star$, we can obtain an expression for $N(x)$ and $\vec{N}(x)$ at every point in spacetime.

Conservation Law

The number density and number current density must satisfy

$$\frac{\partial N}{\partial t} + \vec{\nabla} \cdot \vec{N} = 0 \quad (\text{or } \partial_\alpha N^\alpha = 0) \tag{5.6}$$

Where the second expression holds in rectangular coordinates. Integrating over a volume $\mathcal{V}$ of space, we obtain:

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathcal{V}} N d^3x}_{\textbf{change in number}} + \underbrace{\int_{\partial\mathcal{V}} \vec{N} \cdot d\vec{A}}_{\textbf{flux through surface}} = 0 \tag{5.7}$$

Conservation of Particles

Summary: Densities of scalar quantities (number, charge) are the time component of a 4-vector. Corresponding spatial components are the current density.

Alternative view: Given a 4-vector number current $\mathbf{N}$

1. The number of particles in volume $\Delta\mathcal{V}$ (spacelike surface), with timelike normal $\mathbf{n}$ is given by $\Delta N = \mathbf{N}\cdot\mathbf{n}\Delta\mathcal{V}$.
2. The flux through area ΔA in time Δt (timelike surface), with spacelike normal $\mathbf{n}$ is given by $\Delta N = \mathbf{N}\cdot\mathbf{n}\Delta\mathcal{A}\Delta t$.

Densities are fluxes in timelike dimensions through spacelike surfaces, currents are fluxes in spacelike dimensions through timelike surfaces.

Energy & Momentum Density

- Densities of energy and momentum are the sources of space-time curvature
- We have introduced a density current 4-vector to associate a scalar quantity with $n_\alpha \Delta \mathcal{V}$
- Energy and momentum form a 4-vector p^α. We need a rank-2 tensor to associate energy/momentum with $n_\alpha \Delta \mathcal{V}$.

Stress Energy Tensor: $T^{\alpha\beta}$

$$\Delta p^\alpha = T^{\alpha\beta} n_\beta \Delta \mathcal{V} \tag{5.8}$$

Energy & Momentum Density

Consider a space-like volume ΔV with a timelike normal $n^\alpha = (1, 0, 0, 0)$. Then

- Energy density is

$$\epsilon = \frac{\Delta p^t}{\Delta \mathcal{V}} = T^{tt} \tag{5.9}$$

- Momentum density in direction i is

$$\pi^i = \frac{\Delta p^i}{\Delta \mathcal{V}} = T^{it} \tag{5.10}$$

Example: Box of particles

Box of particles of rest mass m, number density n. With respect to a moving frame:

1. energy $= m\gamma$ where $\gamma = (1 - V^2)^{-1/2}$
2. number density $N = n\gamma$.

Energy density is number density $\times$ particle energy

$$\epsilon \equiv T^{tt} = (m\gamma)(n\gamma) = mnu^t u^t$$

Momentum density is number density $\times$ particle momentum

$$\pi^i \equiv T^{it} = (m\gamma V^i)(n\gamma) = mnu^i u^t$$

The components of the stress-energy tensor are

$$T^{\alpha\beta} = mnu^\alpha u^\beta$$

Interpretation of $T^{\alpha j}$

Consider a 3-volume spanned by $\Delta y, \Delta z, \Delta t$ (normal $n = (0, 1, 0, 0)$). Then the associated momentum is

$$\Delta p^\alpha = T^{\alpha x} \Delta y \Delta z \Delta t$$

Time component T^{tx}

$$T^{tx} = \frac{\Delta p^t}{\Delta A \Delta t}$$

Flux of energy in the x-direction.

Flux of energy is the same thing as a momentum density.

Example: Box of particles.

For each particle, $E = m\gamma$, $\vec{p} = m\gamma \vec{V}$. So energy flux = (energy density) $\times$ V = momentum density.

Interpretation of $T^{\alpha j}$

Spatial components T^{ix}

$$T^{ix} = \frac{\Delta p^i/\Delta t}{\Delta A}$$

i-th component of force per unit area exerted across a surface with normal in x-direction Flux of energy in the x-direction.

More generally, components of force $\vec{F}$ exerted across area ΔA with normal $\vec{n}$:

$$\Delta F^i = T^{ij} n_j \Delta A$$

T^{ij} is the i-th component of force per unit area exerted across a surface with normal in j-direction

Example: Pressure

For a fluid at rest, the force exerted across a surface is always along its normal, and the same for all directions.
The stress tensor is diagonal, with diagonal entries equal to pressure p:

$$T^{ij} = p\delta^{ij}$$

Stress Energy Tensor

$$T^{\alpha\beta} = \left(\begin{array}{c|c} \text{energy} & \text{energy} \\ \text{density} & \text{flux} \\ \hline \text{mom.} & \text{stress} \\ \text{density} & \text{tensor} \end{array} \right) \tag{5.11}$$

The stress energy tensor is symmetric, $T^{\alpha\beta} = T^{\beta\alpha}$.
Momentum density is equivalent to energy flux.

Example: Energy density measured by an Observer

Consider an observer with 4-velocity $\mathbf{u}_{\rm obs}$, and a small volume $\Delta\mathcal{V}$ with unit normal $\mathbf{n} = -\mathbf{u}_{\rm obs}$.
Energy momentum of the matter contained in that volume is

$$\Delta p_\alpha = T_{\alpha\beta}(-u^\beta_{\rm obs})\Delta\mathcal{V}$$

So the energy is

$$\Delta E = -\Delta\mathbf{p}\cdot\mathbf{u}_{\rm obs} = T_{\alpha\beta}u^\alpha_{\rm obs}u^\beta_{\rm obs}\Delta\mathcal{V}$$

Energy density:

$$T_{\alpha\beta}u^\alpha_{\rm obs}u^\beta_{\rm obs}$$

or $T_{\hat{0}\hat{0}}$ in orthonormal frame where $\mathbf{e}_{\hat{0}} = \mathbf{u}_{\rm obs}$.

Conservation of Energy–Momentum

Conservation of Energy-Momentum in flat space-time

In flat space, energy of matter is conserved, so is the momentum of matter. These four conservation equations can be written as

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} = 0 \tag{5.12}$$

Time component:

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{\pi} = 0$$

where ϵ is the energy density, $\vec{\pi}$ is the momentum density or energy flux. Similar to number density equation we saw earlier.

Conservation of Energy–Momentum

Spatial components:

$$\frac{\partial \pi^i}{\partial t} = -\frac{\partial T^{ij}}{\partial x^j} \equiv \phi^i$$

where ϕ^i is a force density. This is $\vec{F} = m\vec{a}$ in continuum.

Example: Pressure force on a Volume

Consider a cube of size L (with edges aligned with the axes) in a fluid for which $T^{ij} = \delta^{ij} p$. Pressure forces on the $y - z$ faces are:

$$F_1 = L^2 p(x, y, z)$$
$$F_2 = -L^2 p(x + L, y, z)$$

The net force is

$$F^x = F_1 - F_2 = L^2[p(x, y, z) - p(x + L, y, z)]$$
$$\approx -L^3 \frac{\partial p}{\partial x}$$

This agrees with our definition of force density ($\phi = F/L^3$):

$$\phi^x = -\frac{\partial T^{xj}}{\partial x^j} = -\frac{\partial p}{\partial x}$$

Perfect Fluid

A fluid is "perfect" when we can ignore heat condition, viscosity, transport and dissipative processes. In this case:

$$\begin{aligned} T^{\alpha\beta} &= \mathrm{diag}(\rho, p, p, p) && \text{[Fluid rest frame]} \\ &= (\rho + p)u^{\alpha}u^{\beta} + \eta_{\alpha\beta}p && \text{[General expression]} \end{aligned}$$

Here **u** is the fluid 4-velocity, ρ is the energy density and p is the pressure. The energy density and pressure are related by an equation of state:

$$\begin{aligned} p &= 0 && \text{dust} \\ p &= \frac{\rho}{3} && \text{CMBR} \\ p &= -\rho && \text{vacuum energy} \end{aligned}$$

Conservation of Energy-Momentum

We want to generalise all of this to curved spacetime:

Perfect fluid stress energy

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + g_{\alpha\beta} p \tag{5.13}$$

in curved spacetime

Local conservation of Energy Momentum

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{5.14}$$

Energy of matter is not conserved in curved spacetime, but changes in response to the curved geometry.

Exercise 12

Show that the stress energy of the vacuum:

$$T^{\alpha\beta}_{\mathrm{vac}} = -\rho_{\mathrm{vac}} g^{\alpha\beta} = -\frac{\Lambda}{8\pi G} g^{\alpha\beta}$$

satisfies the local conservation law

$$\nabla_\beta T^{\alpha\beta} = 0$$

provided that Λ is a constant.

The Einstein Equation

The Einstein Equation

$$\begin{pmatrix}\text{a measure of local} \\ \text{space-time curvature}\end{pmatrix} = \begin{pmatrix}\text{a measure of matter} \\ \text{energy density}\end{pmatrix} \quad (5.15)$$

We now have candidates for both curvature ($R_{\alpha\beta}$, R) and energy density $T_{\alpha\beta}$. This suggests an equation of the form

$$R_{\alpha\beta} + \lambda g_{\alpha\beta} R = \kappa T_{\alpha\beta}$$

for some constants λ and κ.

The Einstein Equation: Fixing λ

We know that $\nabla_\beta T^{\alpha\beta} = 0$. We use this to fix the value of λ.To do this, we need the:

Bianchi Identity

$$\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0$$

By contracting indices appropriately, we obtain

$$\nabla_\beta (R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R) = 0 \tag{5.16}$$

This fixes $\lambda = -\frac{1}{2}$. Introduce Einstein Tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \tag{5.17}$$

The Einstein Equation: Fixing κ

We fix κ by requiring that the Einstein equation reduces to Newton's gravity in the weak field limit:

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\left[dx^2 + dy^2 + dz^2\right]$$

In weak field, low velocity limit:

- Rest mass energy, μ is small
- Kinetic energy μV^2 is smaller (as $V \ll c$)
- Potential energy $\mu\Phi$ is smaller (Φ is small)

So, only significant component of stress energy $T^{\alpha\beta}$ is

$$T^{tt} = \mu + \ldots \qquad (5.18)$$

The Einstein Equation: Fixing κ

In exercise 10, we calculated $R_{jtkt} = \partial_j \partial_k \Phi$. So, $R_{tt} = \nabla^2 \Phi$. You can calculate the rest of the components of the curvature tensor in a similar way (Appendix B of Hartle), and obtain:

$$G_{tt} = 2\nabla^2 \Phi + \ldots$$

and all other components of $G_{\alpha\beta}$ are of order Φ^2.
So, we have

$$2\nabla^2 \Phi = \kappa \mu$$

Comparing with Newtonian gravity ($\nabla^2 \Phi = 4\pi G \mu$) fixes $\kappa = 8\pi G$.

The Einstein Equation

Putting it all together, we get:

The Einstein Equation

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta} \tag{5.19}$$

Lectures 13-15: Gravitational Wave Emission [Hartle, Ch. 23]

Linearized Einstein Equation with Sources

We now want to study solutions to the linearised Einstein equation in the presence of matter. As before, we expand the metric as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \text{where} \quad |h_{\alpha\beta}| \ll 1 \tag{6.1}$$

and look for a solution to

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta} \tag{6.2}$$

where the curvature produced by the source $T_{\alpha\beta}$ is weak enough that the linearised approximation holds.

"Trace-reversed" amplitude

When solving the linearised equations in vacuum, it was useful to introduce the Lorentz condition

$$V_\alpha = \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta = 0$$

This is much simpler if we introduce the "trace-reversed" amplitude

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h \qquad (6.3)$$

Then, the Lorentz condition simplifies to

$$\partial_\beta \bar{h}^{\alpha\beta} = 0 \qquad (6.4)$$

Linearized Einstein Equation with Sources

We showed before (4.15) that in the Lorentz gauge,

$$\delta R_{\alpha\beta} = -\frac{1}{2}\Box h_{\alpha\beta}$$

From which, we get:

$$G_{\alpha\beta} = \delta R_{\alpha\beta} - \frac{1}{2}\delta R = -\frac{1}{2}\Box \bar{h}_{\alpha\beta} \qquad (6.5)$$

So that

$$\Box \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta} \qquad (6.6)$$

Solving the wave equation with a point source

Let's start with the scalar wave equation:

$$-\frac{\partial^2 f(x)}{\partial t^2} + \vec{\nabla}^2 f(x) = j(x). \tag{6.7}$$

Start by solving for a source localised in space and time, i.e.

$$j(x) = \delta(t)\delta(x)\delta(y)\delta(z) \tag{6.8}$$

Spherical symmetry implies the solution can depend only on t and $r = |x|$. Away from the source, the solution is

$$g(t, r) = \frac{1}{r}\left[O(t - r) + I(t + r)\right] \tag{6.9}$$

which represent Outgoing and Ingoing waves. Physically, we are only interested in outgoing waves.

Solving the wave equation with a point source

The solution to

$$-\frac{\partial^2 g(x)}{\partial t^2} + \vec{\nabla}^2 g(x) = \delta(t)\delta(x)\delta(y)\delta(z) \qquad (6.10)$$

is

$$g(t, r) = -\frac{\delta(t-r)}{4\pi r} \qquad (6.11)$$

Can show this by integrating over a small volume near the origin.

Solving the wave equation

Since the wave equation is *linear*, the solution for a general source is the sum of the solutions for many point sources. The solution to the wave equation with source $j(t,x)$ can be written:

$$f(t,x) = \int dt' d^3x' g(t-t', \vec{x}-\vec{x}')j(t',x') \qquad (6.12)$$

Since $g(t-t',\vec{x}-\vec{x}') \propto \delta(t-t'-|\vec{x}-\vec{x}'|)$, we evaluate the integral at the retarded time, $t' = t_{\mathrm{ret}} \equiv t - |\vec{x}-\vec{x}'|$:

$$f(t,x) = -\frac{1}{4\pi}\int d^3x' \frac{[j(t',x')]_{\mathrm{ret}}}{|\vec{x}-\vec{x}'|} \qquad (6.13)$$

Linearized Gravity Solution

Returning to linearised gravity, the solution is

$$\bar{h}^{\alpha\beta}(t,\vec{x}) = 4\int d^3x' \frac{\left[T^{\alpha\beta}(t',\vec{x}')\right]_{\mathrm{ret}}}{|\vec{x}-\vec{x}'|} \qquad (6.14)$$

Note: this necessarily satisfies the Lorentz gauge condition. Finally, if we are far from the source ($r \gg R_{\mathrm{source}}$) and the waves have a long wavelength ($\lambda \gg R_{\mathrm{source}}$), then

$$\bar{h}^{\alpha\beta}(t,\vec{x}) \xrightarrow[r\to\infty]{} \frac{4}{r}\int d^3x'\, T^{\alpha\beta}(t-r,\vec{x}') \qquad (6.15)$$

We'd like to rewrite in terms of $T^{tt} = \mu$, the mass density.

Exercise 13

1. From the conservation law for $T^{\alpha\beta}$:

$$\partial_t T^{tt} + \partial_k T^{kt} = 0 \quad \text{and} \quad \partial_t T^{tk} + \partial_l T^{lk} = 0\,,$$

show that

$$\frac{\partial^2 T^{tt}}{\partial t^2} = \frac{\partial^2 T^{kl}}{\partial x^k \partial x^l}$$

2. By Multiplying by $x^i x^j$ and integrating over a volume that encompasses the source (and integrating by parts) show

$$\int d^3x T^{ij}(x) = \frac{1}{2}\frac{d^2}{dt^2}\int d^3x\, x^i x^j T^{tt}(x)$$

Generation of GWs

For systems with weak gravitational fields, long wavelength, large r, the metric perturbation far from source is

$$\bar{h}^{ij}(t,\vec{x}) = \frac{2}{r}\,\dddot{I}^{ij}(t-r) \qquad (6.16)$$

Here $\bar{h}^{ij}$ is the "trace-reversed" perturbation:

$$\bar{h}^{ij} \equiv h^{ij} - \frac{1}{2}\,\delta^{ij}\,h^k_k\,, \qquad (6.17)$$

I^{ij} is the "second mass moment" of the source:

$$I^{ij}(t) \equiv \int d^3x\,\mu(t,\vec{x})\,x^i x^j\,, \qquad (6.18)$$

μ is the mass-density of the source, and the dot $(\dot{\ })$ is $\partial/\partial t$.

Example: Binary Systems

Consider two stars of the same mass M in orbit of radius R around their centre of mass, with a period P.
Order-of-magnitude estimate of GW amplitude:

$$I \sim 2MR^2$$
$$\ddot{I} \sim 2MR^2/P^2$$

For circular motion

$$\frac{M}{(2R)^2} = \frac{V^2}{R} = \frac{(2\pi R/P)^2}{R}$$

So,

$$\bar{h}^{ij} \sim \left(\frac{M}{r}\right)\left(\frac{M}{R}\right) \quad \text{or} \quad \bar{h}^{ij} \sim \left(\frac{M}{r}\right)\left(\frac{M}{P}\right)^{2/3} \qquad (6.19)$$

Exercise 14

1. For a neutron-star binary ($M \simeq 1.4 M_\odot$) at 6 kpc with $P = 8$ hr show that $h \sim 10^{-23}$.
2. What is the period when the system is about to merge (Neutron stars have a radius of about 10 km)?
3. What is the gravitational wave amplitude at earth at merger?

Gravitational Radiation from Binary Stars

Let's do the calculation in detail. From

$$\frac{M}{(2R)^2} = \frac{V^2}{R} = \frac{(2\pi R/P)^2}{R}$$

and $\Omega = \frac{2\pi}{P}$ we get

$$R = \left(\frac{M}{4\Omega^2}\right)^{1/3} = \left(\frac{MP^2}{16\pi^2}\right)^{1/3}$$

First, check we are in the long wavelength, large distance limit. $\lambda \sim \frac{2\pi}{\Omega}$, so $\frac{R}{\lambda} \lesssim \left(\frac{M}{R_\star}\right)^{1/2}$. And $R_\star \geq 2M$, so this is satisfied.

Gravitational Radiation from Binary Stars

For an equal mass binary, the position of one of the stars is

$$x(t) = R\cos\Omega t\,, \quad y(t) = R\sin\Omega t\,, \quad z(t) = 0$$

The second mass moment is

$$\begin{aligned} I^{xx} &= 2MR^2\cos^2\Omega t = MR^2\left[1+\cos(2\Omega t)\right] \\ I^{xy} &= 2MR^2\sin\Omega t\cos\Omega t = MR^2\sin(2\Omega t) \\ I^{xx} &= 2MR^2\sin^2\Omega t = MR^2\left[1-\cos(2\Omega t)\right] \end{aligned}$$

and all other components vanish.

Gravitational Radiation from Binary Stars

Substituting into

$$\bar{h}^{ij}(t,\vec{x}) = \frac{2}{r}\,\ddot{I}^{ij}(t-r) \tag{6.20}$$

gives

$$\bar{h}^{ij} = -\frac{8\Omega^2 MR^2}{r}\begin{pmatrix} \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.21}$$

Gravitational wave frequency is twice orbital frequency.

Directional dependence

1. For an observer in the z-direction, the perturbation is transverse-traceless. It constitutes two polarisations h_+ and $h_\times$ of equal amplitude but 90° out of phase.
2. For an observer in the x-direction, calculate transverse-traceless perturbation by:
   - setting longitudinal terms to zero
   - subtracting off trace, to get

$$\bar{h}^{ij} = -\frac{4\Omega^2 MR^2}{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\cos(2\Omega t) & 0 \\ 0 & 0 & \cos(2\Omega t) \end{pmatrix} \quad (6.22)$$

One polarisation and half the amplitude.

Energy Loss in Gravitational Waves

Motivate the equation for energy in gravitational waves from:

1. The argument that energy in gravitational waves should be proportional to the amplitude $\bar{h}^{ij}$ squared.
2. Luminosity (or power) is dimensionless (in geometric units). Only the third time derivative of I^{ij} is dimensionless.
3. Luminosity is a scalar, so it can only depend on $\dddot{I}_{ij}\dddot{I}^{ij}$ and $(\dddot{I}^{k}{}_{k})^2$.

This leads to (we have not derived the numerical factors):

$$L_{\mathrm{GW}} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I\!\!\!-}^{ij}\dddot{I\!\!\!-}_{ij}\rangle \quad \text{where} \quad I\!\!\!-^{ij} = I^{ij} - \frac{1}{3}\delta^{ij} I^{k}{}_{k}$$

and $\langle\cdot\rangle$ denotes average over a period.

Energy loss from a binary

Using the previous expression for I^{ij}, and recalling that $\langle \sin^2 2\Omega t \rangle = \langle \cos^2 2\Omega t \rangle = \frac{1}{2}$, we get

$$L_{\mathrm{GW}} = \frac{128}{5} M^2 R^4 \Omega^6$$

for an equal mass binary system.
Using Kepler's law, and inserting G and c factors gives

$$L_{\mathrm{GW}} = 1.9 \times 10^{33} \left(\frac{M}{M_\odot} \frac{1\mathrm{h}}{P} \right)^{10/3} \mathrm{erg/s} \qquad (6.23)$$

For comparison, solar luminosity is $L_\odot = 3.9 \times 10^{33} \mathrm{erg/s}$.

Evolution of the orbit

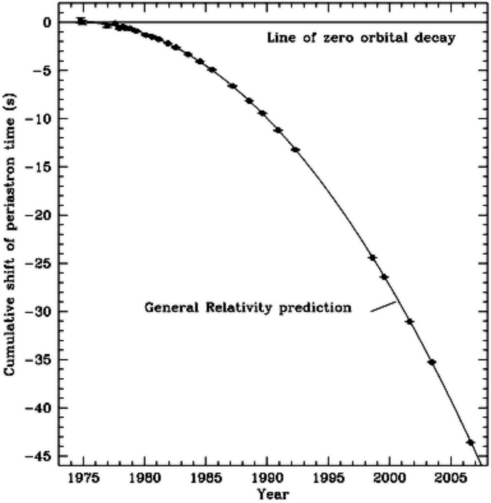
The energy of a binary system, in Netwonian mechanics, is

$$E = KE + PE = 2\left(\frac{1}{2}MV^2\right) - \frac{M^2}{2R^2}$$
$$= -\frac{M^2}{4R^2} = -\frac{M}{4}\left(\frac{4\pi M}{P}\right)^{2/3}$$

Energy is negative (bound system), so radiating energy causes orbit to shrink.
Rate of change of energy is $\frac{dE}{dt} = -L_{\mathrm{GW}}$, giving

$$\frac{dP}{dt} = -3.4 \times 10^{-12}\left(\frac{M}{M_\odot}\frac{1\mathrm{h}}{P}\right)^{5/3} \qquad (6.24)$$

Hulse Taylor Pulsar



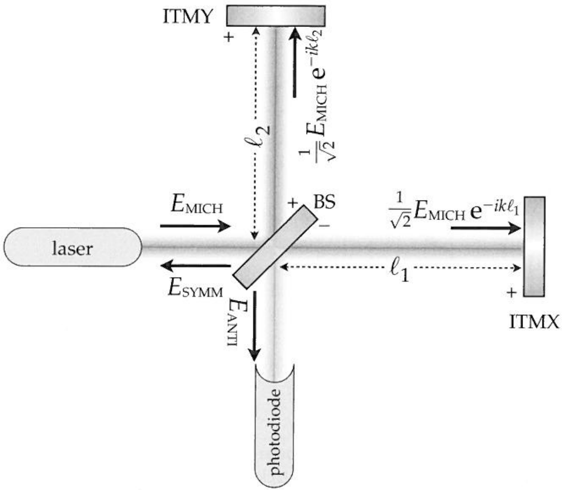
- Masses $1.4 M_\odot$
- Period 7.75 hours.
- Orbit decreases by $\sim 10\mu s$ per year.
- Observed decay matches GR prediction (incorporating ellipticity)

Lectures 16-17: Gravitational Wave Detectors [Maggiore Ch 9; Creighton/Anderson Ch 6]

How sensitive?

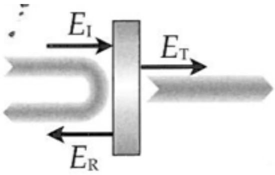
- We have shown (Exercise 14) that the gravitational wave strain from the Hulse-Taylor pulsar will be about $h \sim 10^{-18}$ at merger.
- But, we'd have to wait about 300 million years for this.
- Alternatively, we could detect any merger in the galaxy. These are expected to happen about once every $10,000$ years (as high as 1 per thousand, as low as 1 per million years).
- Instead, look for a mergers in a million galaxies. Galactic density is about 0.01 Milky Way equivalent galaxies per Mpc^3. i.e. build a detector sensitive to binary mergers out to $\sim$ 100 Mpc.
- This requires a sensitivity of about $h \sim 10^{-23}$.

A basic Michelson interferometer



A basic Michelson interferometer

For a lossless mirror:



$$E_T = tE_I \quad E_R = rE_I$$
$$|t|^2 + |r|^2 = 1$$

For light incident from the other side, $t' = t$, $r' = -r$.

For the beam splitter, half of the light passes through:

$$r = t = \frac{1}{\sqrt{2}}.$$

When the arms are equal length, there is destructive interference and no light is seen in the photodiode. [The fact that $r' = -r$ explains why we have destructive interference.]

Estimate of sensitivity

Change in length $\Delta\ell = \Delta\ell_1 - \Delta\ell_2$. Measurable change in length when $\Delta\ell \approx \lambda_{\rm laser}$.
Assume a kilometre scale detector, and $\lambda_{\rm laser} \sim 1\mu m$. Then, measurable GW strain:

$$h = \frac{\Delta\ell}{\ell} \sim \frac{\lambda_{\rm laser}}{\ell} \sim \frac{10^{-6}{\rm m}}{10^{3}{\rm m}} = 10^{-9}$$

This is 14 orders of magnitude too large! This can be overcome by using longer effective arms, better measurement of $\Delta\ell$ and high laser power.

Increasing the arm length

We can increase the effective arm length by making the light travel up and down the arm numerous times. The longest feasible is

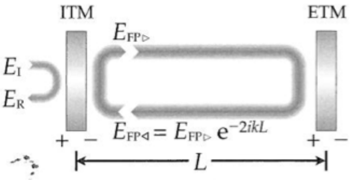
$$\ell_{\mathrm{eff}} \sim \lambda_{\mathrm{GW}} = c/f_{\mathrm{GW}}$$

For ground based detectors, $f_{\mathrm{GW}} = 100 - 1000\mathrm{Hz} \Rightarrow \ell_{\mathrm{eff}} \sim 1000\mathrm{km}$. This gets us to

$$h = \frac{\Delta \ell}{\ell_{\mathrm{eff}}} \sim \frac{\lambda_{\mathrm{laser}}}{\lambda_{\mathrm{GW}}} \sim \frac{10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-12}$$

Fabry Perot cavities

Comprised of two mirrors: one partially transmissive, one fully reflective:



$$E_{FP\triangleright} = t_{ITM} E_I - r_{ITM} E_{FP\triangleleft}$$
$$= t_{ITM} E_I - r_{ITM} e^{-2ikL} E_{FP\triangleright}$$

This gives

$$E_{FP\triangleright} = \frac{t_{ITM}}{1 + r_{ITM} e^{-2ikL}} E_I$$

At resonance, $e^{-2ikL} = -1$ so

$$E_{FP\triangleright} = \frac{t_{ITM}}{1 - r_{ITM}} E_I \,.$$

so if r_{ITM} is close to 1, can build up a lot of power. Light does (on average) $\frac{1}{1-r_{ITM}^2}$ round trips before coming out.

Response to a gravitational wave

For a simple Michelson interferometer,

1. At low frequencies, $\Delta t \propto h\ell/c$
2. The detector will be insensitive to GW with $\ell = \lambda_{\rm GW}$.

In general, the response is proportional to $h\ell\,\mathrm{sinc}(\ell/\lambda_{\rm GW})$.

More complicated for a Fabry-Perot cavity as light undergoes varying number of round trips, but still lose sensitivity at higher frequencies.

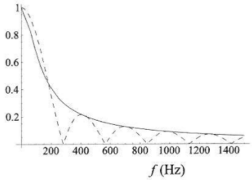


Fig. 9.13 A plot of the function $[1+(f/f_p)^2]^{-1/2}$, (solid line), compared to the function $|\mathrm{sinc}\,(f/f_p)|$ (dashed line). We have taken $f_p =$ 90 Hz.

Dashed: Michelson
Solid: Fabry Perot.

Improved measurement of the optical fringe

Can measure length changes much smaller than $\lambda_{\rm laser}$, what's the limit? Want to measure light intensity at the photodiode. Random fluctuations in the number of observed photons are $\sim N_{\rm photons}^{1/2}$, can't measure physical changes smaller than this, i.e.

$$\Delta\ell \sim \frac{N_{\rm photons}^{1/2}}{N_{\rm photons}} \lambda_{\rm laser}$$

We can only collect photons for a time about equal to the period of the gravitational wave, i.e. $\tau \sim 1/f_{\rm GW}$. Total number of photons collected depends upon GW frequency and laser power:

$$N_{\rm photons} = \frac{P_{\rm laser}}{hc/\lambda_{\rm laser}} \tau \sim \frac{P_{\rm laser}\lambda_{\rm laser}}{hc\, f_{\rm GW}}$$

Exercise 15

1. Using

$$N_{\rm photons} = \frac{P_{\rm laser}}{hc/\lambda_{\rm laser}}\tau \sim \frac{P_{\rm laser}\lambda_{\rm laser}}{hc\, f_{\rm GW}}$$

   and taking

$$P_{\rm laser} = 1W$$
$$\lambda_{\rm laser} = 1\mu m$$
$$f_{\rm GW} \sim 300\rm Hz$$

   estimate the number of photons that can be collected.

2. What is the sensitivity $h = \frac{\delta\ell}{\ell_{\rm eff}}$ that can be achieved?

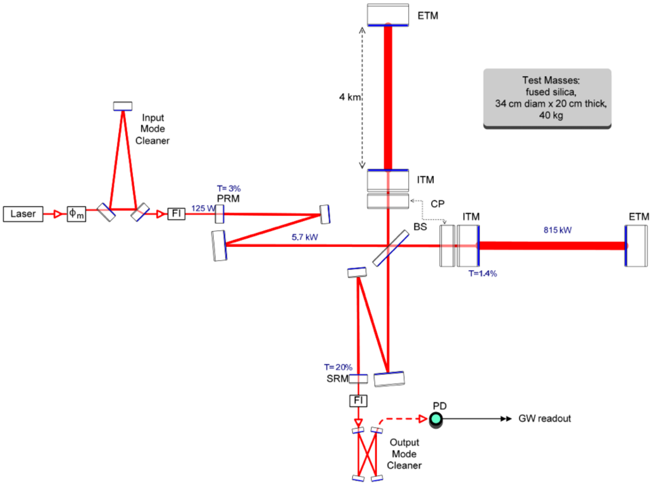
Higher laser power & signal recycling

We saw that with a $1W$ laser, can get a sensitivity $h \sim 10^{-20}$. Can do better with higher laser power. Achieved in two ways:

1. *Use a higher power input laser.* Advanced LIGO uses $150W$.
2. *Use power recycling.* When the interferometer arms are equal length, there is no signal at the photodiode — it all goes back out towards the input. Simply reflect this back into the detector to build up laser power

In a similar way, can use *signal recycling.* Reflect the signal that comes out of the interferometer back into the detector. This can improve the sensitivity of the detector.

The advanced LIGO detector



The LIGO Hanford Observatory

Limits to detector sensitivity

We have argued that it's possible, in principle, to make a gravitational wave detector that is as sensitive as required. Let's look now at some of the sources of noise that limit the sensitivity. We will discuss

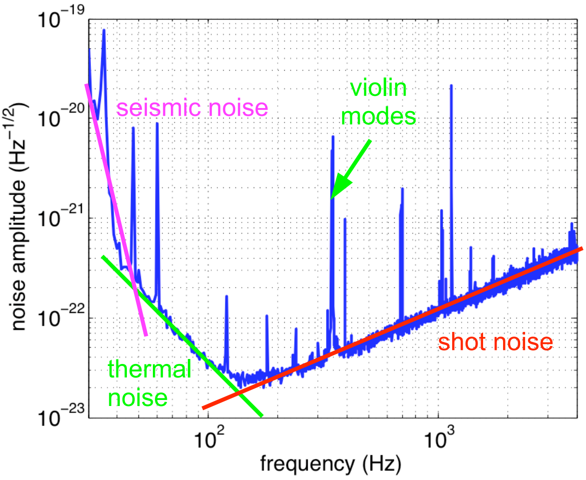
- Quantum noise: shot noise and radiation pressure.
- Thermal noise
- Seismic noise

Characterize sensitivity using the noise spectral density

$$\frac{1}{2}S_h(f) = \langle|\tilde{n}(f)|^2\rangle\Delta f$$

or amplitude strain sensitivity $S_h^{1/2}(f)$.

LIGO-Hanford Detector Noise Spectrum, 2007



Shot noise

Shot noise is simply the photon counting noise we met before.

$$\frac{\Delta\ell}{\ell} \sim \frac{\lambda_{\rm laser}}{\ell N_{\rm photons}^{1/2}} = \frac{1}{\ell}\sqrt{\frac{hc\lambda_{\rm laser}}{P\tau}}$$

Detailed calculation for a Michelson interferometer gives (see Maggiore):

$$S_{\rm shot}^{1/2}(f) = \frac{\lambda_{\rm laser}}{4\pi L}\left(\frac{2\hbar\omega_{\rm laser}}{P}\right)^{1/2}$$

Notes:

1. This is frequency independent, but for a realistic (Fabry–Perot) interferometer, need to account for detector response.
2. Higher power reduces shot noise.

Radiation Pressure

Can't increase laser power indefinitely without repercussions.

- Photons bouncing off mirrors impart momentum, moving the mirrors stochastically.
- More power $\longrightarrow$ more photons $\longrightarrow$ more **radiation pressure noise**.

Momentum transfer to mirror due to reflection of a photon is $2|\mathbf{p}|$, and energy is $E_\gamma = |\mathbf{p}|/c$ so, force due to wave with power P is

$$F = 2P/c\,.$$

Fluctuations in time τ are

$$\Delta F = 2\sqrt{\frac{\hbar\omega_{\text{laser}} P}{c^2\tau}}$$

Higher power leads to greater variation in force.

Radiation Pressure

Spectrum of radiation pressure force:

$$F(f) = 2\sqrt{\frac{2\pi\hbar P_{\rm in}}{c\lambda}}\,.$$

Corresponding motion of each mirror (from $F = ma$):

$$x(f) = \frac{F(f)}{m(2\pi f)^2}\,.$$

The power spectrum of the fractional relative length change from the sum of the **2** arms is then

$$S_{\rm rad}^{1/2}(f) = \frac{4}{ML(2\pi f)^2}\sqrt{\frac{2\hbar\omega_{\rm laser}P}{c^2}}$$

Want **lowest laser power** to reduce radiation pressure noise.

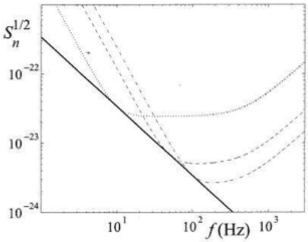
The standard quantum limit

Shot noise decreases with laser power, while radiation pressure increases with laser power.

At any given frequency $f = f_0$ the sum $S_{\rm shot} + S_{\rm rad}$ is minimized by choosing the power P to make $S_{\rm shot} = S_{\rm rad}$. The total noise is then

$$S_{\rm SQL}^{1/2} = \frac{1}{2\pi f L}\sqrt{\frac{8\hbar}{M}}.$$

This minimum noise level is called the **standard quantum limit**.



Radiation pressure and shot noise contributions for different values of f_0.

Seismic Noise

Seismic noise is due to shaking of the ground, due to earthquakes, weather, human activity, etc. It affects the interferometer by shaking the optical components.
If the mirrors were resting on the ground, the seismic noise would be

$$S_{\rm seis}^{1/2} \sim 10^{-12}\mathrm{Hz}^{-1/2}\left(\frac{10\mathrm{Hz}}{f}\right)^2 \quad \text{for } f > 10\mathrm{Hz}$$

This is many orders of magnitude larger than the gravitational wave signal we are trying to measure. Need to suppress the effect of seismic motion on the motion of the mirrors: suspend them as pendular.

Seismic Noise

Mirrors are suspended as pendula (simple harmonic oscillators). If X is the position of the pivot point, x is the position of the mirror and ℓ is the length of the pendulum, then

$$\ell \frac{d^2 x}{dt^2} = -g(x - X)$$

Taking Fourier transform:

$$-4\pi^2 f^2 \ell \tilde{x} = -g(\tilde{x} - \tilde{X})$$

We can rewrite this as

$$\tilde{x}(f) = A(f)\tilde{X}(f) \quad \text{where} \quad A(f) = \frac{1}{1 - (f/f_{\mathrm{pend}})^2}$$

$$\text{and} \quad f_{\mathrm{pend}} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

Seismic Isolation

For $f > f_{\rm pend}$,

$$\tilde{x}(f) \approx -\left(f_{\rm pend}/f\right)^2 \tilde{X}(f)$$

so seismic noise is suppressed.
Increase isolation from ground by "stacking". e.g. Advanced LIGO uses a quadruple pendulum.
LIGO also uses **active isolation**: sensors monitor motion of suspension system; control motors that push back on system to cancel ground motion.
Can get required sensitivity, but not easy, especially at low frequencies.

Advanced LIGO suspension.

Thermal noise

Equipartition Theorem: each degree of freedom of a system in thermodynamic equilibrium at temperature T should have an energy whose expectation value is $\frac{1}{2}k_B T$.

Main physical manifestations of thermal noise:

- Random motion of atoms in mirrors.
- Vibration in wire suspensions of mirrors ("violin modes").
- Swinging of the mirror pendula.

Induces random "jittering" of positions of mirrors.

- E.g., $m = 10kg$ mirror suspended from a $\ell = 1m$ wire:

$$\Delta L_{\rm rms} \simeq \sqrt{\frac{k_B T \ell}{mg}} \simeq 6 \times 10^{-12} m \,.$$

Fluctuation-Dissipation Theorem

For a linear system subject to a force $F_{\rm ext}$, can always write the equation of motion as

$$Y(f)\tilde{F}_{\rm ext}(f) = 2\pi i f \tilde{x}(f) = \tilde{v}(f)$$

Where Y is the *admittance* of the system.
Then, the power spectrum of thermal fluctuations in equilibrium at temperature T is given by

$$S_{\rm therm}(f) = \frac{4k_B T}{(2\pi f)^2}|Re\left[Y(f)\right]|.$$

This applies to **any** linear system in thermodynamic equilibrium.

Exercise 16

Consider a harmonic oscillator with mass m, spring constant k, and dissipative force $-bv$:

$$F_{\rm ext} = m\ddot{x} + b\dot{x} + kx\,,$$

The resonant frequency is $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

1. Show that the equation of motion can be written as

$$\tilde{F}_{\rm ext}(f) = -4\pi^2 m(f^2 - f_0^2)\tilde{x}(f) + b(2\pi i f)\tilde{x}(f)$$

2. From this, show that the admittance Y is

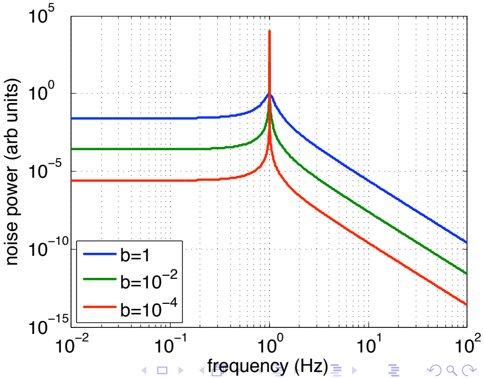
$$Y(f) = \frac{2\pi i f}{2\pi i b f - 4\pi^2 m(f^2 - f_0^2)} = \frac{f^2 b - 2\pi i f m(f^2 - f_0^2)}{b^2 f^2 + (2\pi m)^2[f^2 - f_0^2]^2}$$

Fluctuation-Dissipation Theorem

For this example

$$S_{\text{therm}}(f) = \frac{k_B T}{\pi^2} \frac{b}{b^2 f^2 + (2\pi m)^2 [f^2 - f_0^2]^2} .$$

Total noise power (or RMS displacement) does not depend on the magnitude b of the dissipation, but the **shape** of the power spectrum does depend on b.



Fluctuation-Dissipation Theorem

Can minimize impact by making dissipation (b) small. Concentrates noise power at resonant frequency f_0.

- Suspend mirrors as pendulums (low b, resonant frequency $\sim 1Hz$ – well below observation band).
- Use high-quality-factor wires for suspensions, so vibrational motion of "violin modes" is concentrated in narrow frequency range.
- Minimize the thermal noise from internal vibrations of the mirrors by making them from material with very low dissipation at acoustic frequencies at room temperature (fused silica).

The advanced LIGO noise budget