An exact treatment of back-reaction in relativistic cosmology

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What is back-reaction in cosmology?

Cosmological back-reaction refers to the difference in large-scale expansion between a universe filled with evenly distributed matter, and a universe filled with clustered (or lumpy) matter.

It is important to understand because:

a) Understanding the errors in precision cosmology requires understanding all of the possible sources of error in our models.

and

b) The discovery of dark energy, and its associated problems, motivates ensuring that we fully understand the models we use to interpret the data.

Top-down vs. bottom-up approaches to cosmological modelling

Top-down involves:

1. First specifying symmetries, or some other properties, that are obeyed (in some statistical sense) on large scales.
2. Finding solutions of Einstein's equations that exhibit those symmetries or properties (usually with continuous matter fields).
3. Including structure by perturbing these solutions.

Bottom-up involves:

1. Considering first the constituent parts of the model being constructed (galaxies, or clusters of galaxies, for example).
2. Solving Einstein's equations in the vicinity of these small parts.
3. Constructing a cosmological model from these building blocks, and/or finding ways of extracting information about the cosmological evolution that emerges on large-scales.

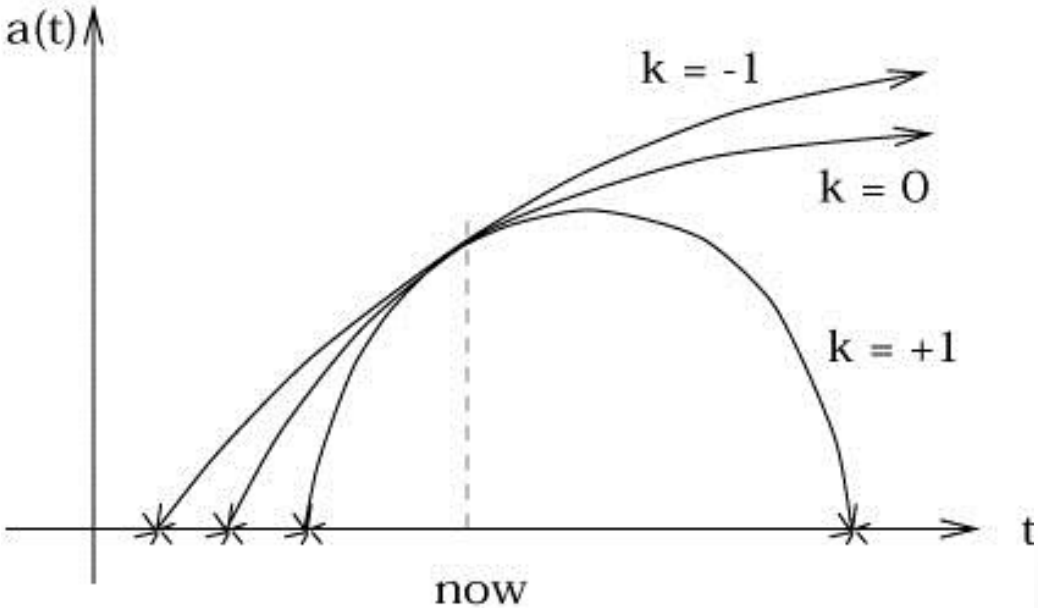
Formalism of top-down approach

- Spatial homogeneity and isotropy imply:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

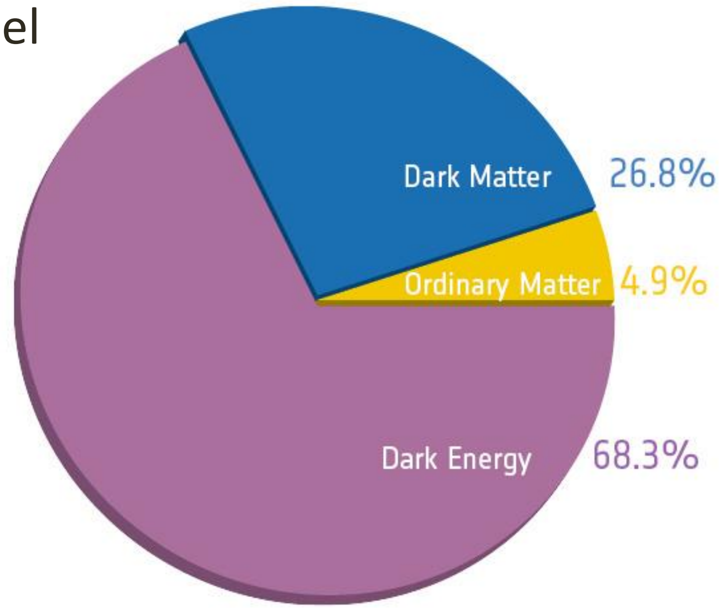
- Einstein's equations give: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$ & $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$

ordinary matter



Results of top-down approach

- The top-down approach leads to a model of the universe that is dominated by its dark components:



- Difficulties for this model:

  i. What are dark matter and dark energy?
  ii. Why do they have the properties they are observed to have?
  iii. How should the homogeneous and isotropic background be chosen?
  iv. Should the evolution of an average geometry obey Einstein's equations?
  v. Is it okay to treat the matter in the Universe as a fluid?

Examples of bottom-up approaches

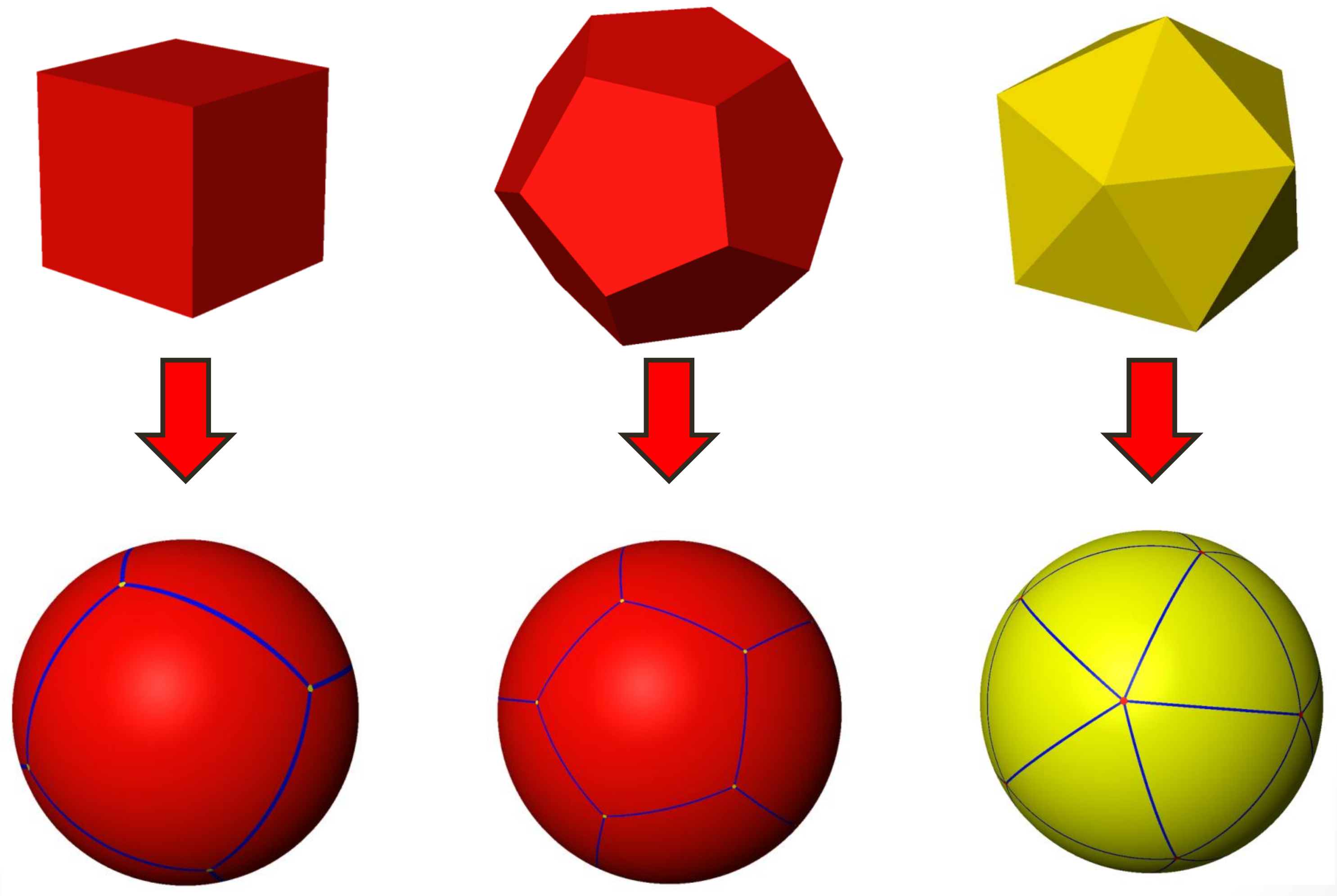
- **Lindquist & Wheeler (1957):** *An approximate, non-perturbative Wigner-Seitz-like construction.*
- **Zalaletdinov (1992) & Buchert (2000):** *Approaches based on averaging geometric quantities and matter fields.*
- **Clifton (2011):** *A model based on matching together different weak-field regions at suitable junctions.*
- **Bentivegna *et al.* (2012) & Yoo *et al.* (2012):** *Application of numerical relativity techniques to cosmology.*
- **Clifton, Gregoris, Rosquist & Tavakol (2013):** *An exact treatment of submanifolds in a universe with discretized matter content.*
- ***Etc. etc.***

Program

The outline of this work is to do the following:

1. Evenly distribute some point-like masses on a topological 3-sphere.
2. Solve the constraint equations on an initial 3-space.
3. Choose some curves in this space that one might consider using to track the large-scale behaviour of the space.
4. Evolve these curves forward in time from the initial hyper-surface.
5. Compare the length of these curves to a Friedmann universe that contains the same total mass.

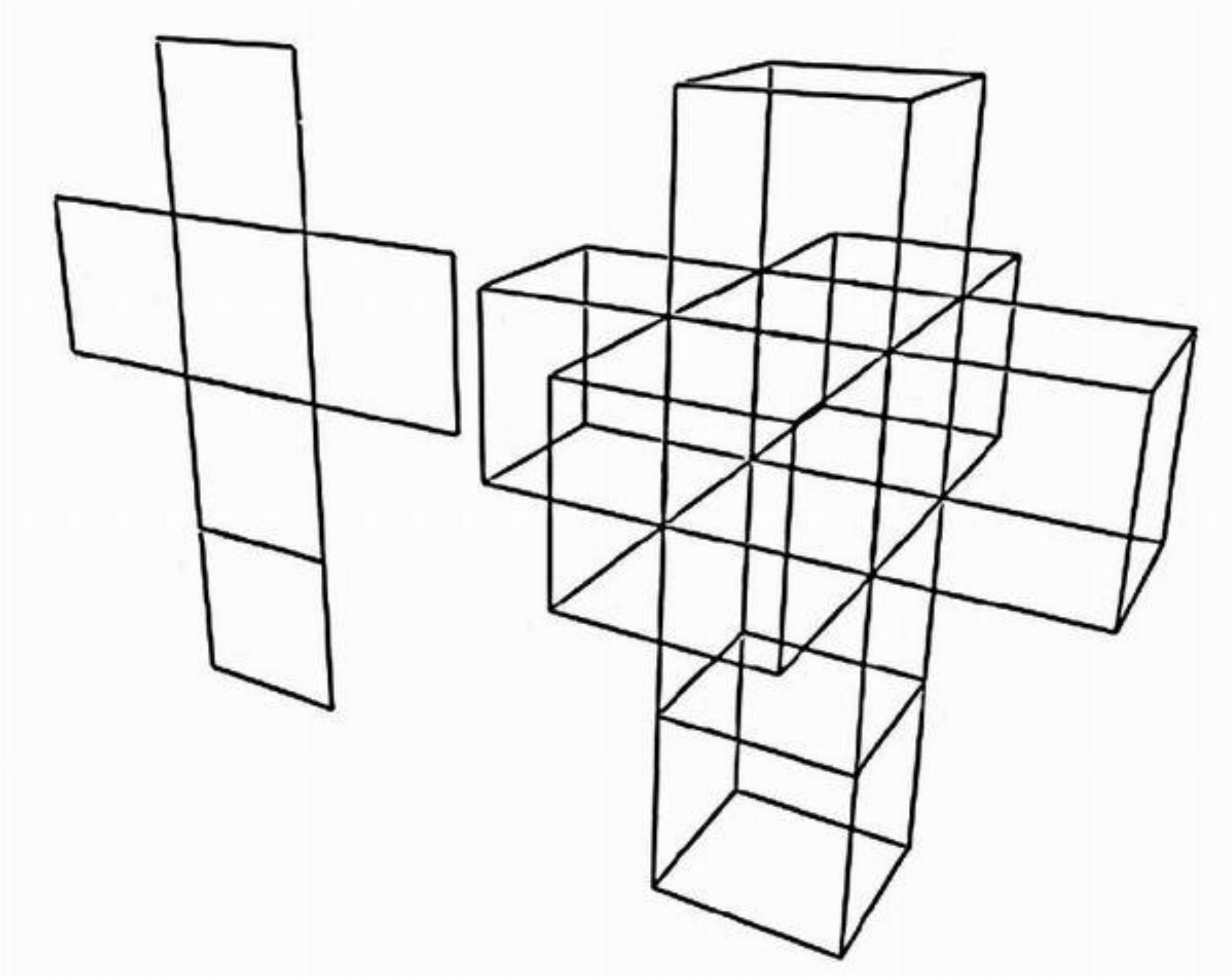
Tiling a sphere (in 2D)



Tiling a sphere (in 3D)

Lattice Type	Number of Faces	Number of Edges	Number of Vertices	Vertex Figure	Schläfli Symbols
5−cell (Tetrahedra)	10 (Triangles)	10	5	Tetrahedron	{333}
8−cell (Cubes)	24 (Squares)	32	16	Tetrahedron	{433}
16−cell (Tetrahedra)	32 (Triangles)	24	8	Octahedron	{334}
24−cell (Octahedra)	96 (Triangles)	96	24	Cube	{343}
120−cell (Dodecahedra)	720 (Pentagons)	1200	600	Tetrahedron	{533}
600−cell (Tetrahedra)	1200 (Triangles)	720	120	Icosahedron	{335}

Example: A (Hyper)-Cube



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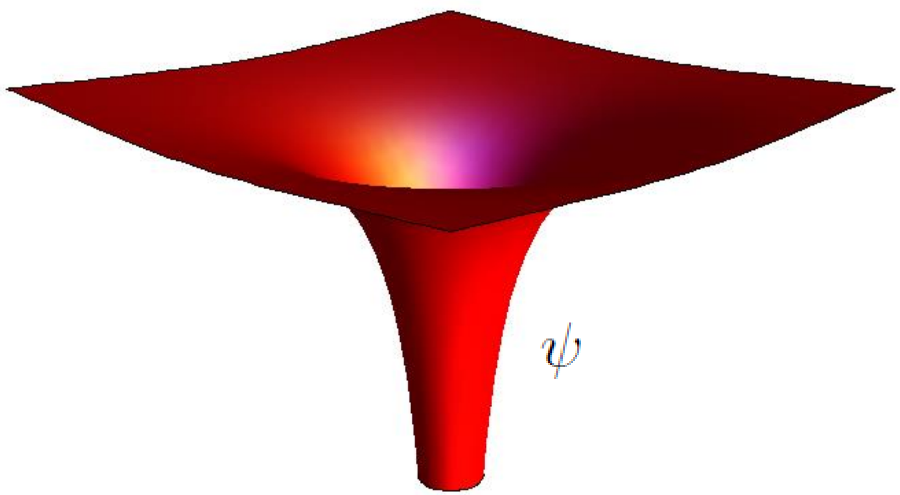
Geometrostatics, in asymptotically flat space

Imposing time-symmetry about an initial hyper-surface gives:

$$dl^2 = \psi^4 \left(dx^2 + dy^2 + dz^2\right)$$

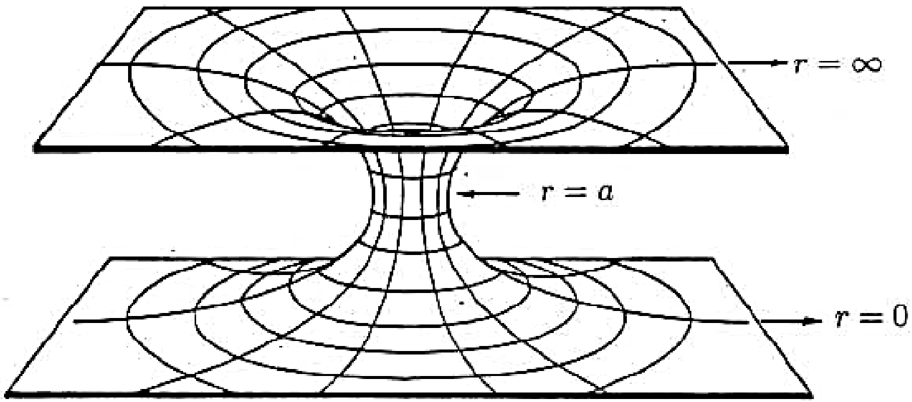
where

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = 0 \quad \Rightarrow \quad \psi = 1 + \sum_i \frac{c_i}{r}$$

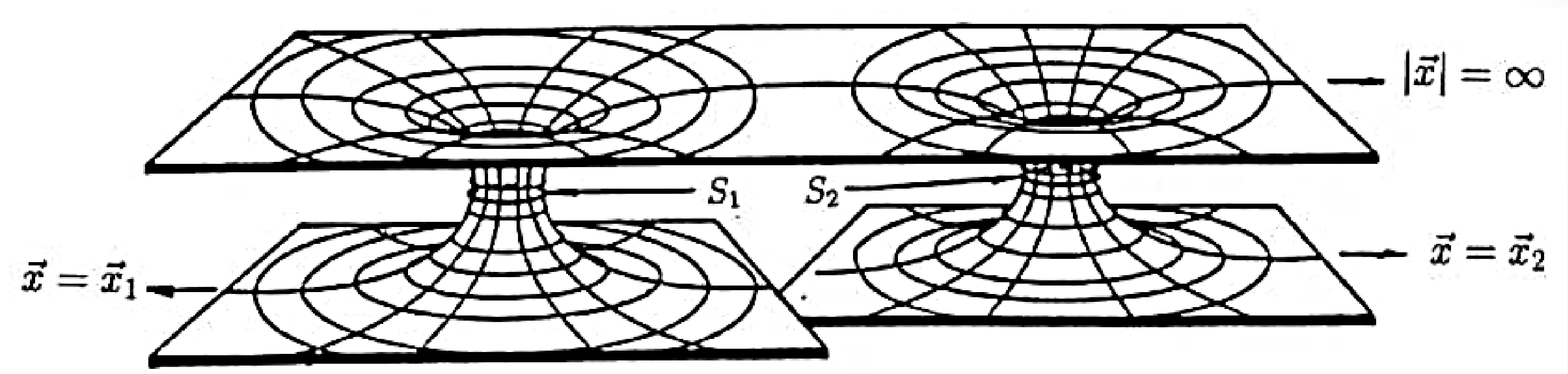


Geometrostatics, in asymptotically flat space

Embedding diagram for one black hole:



For two black holes:

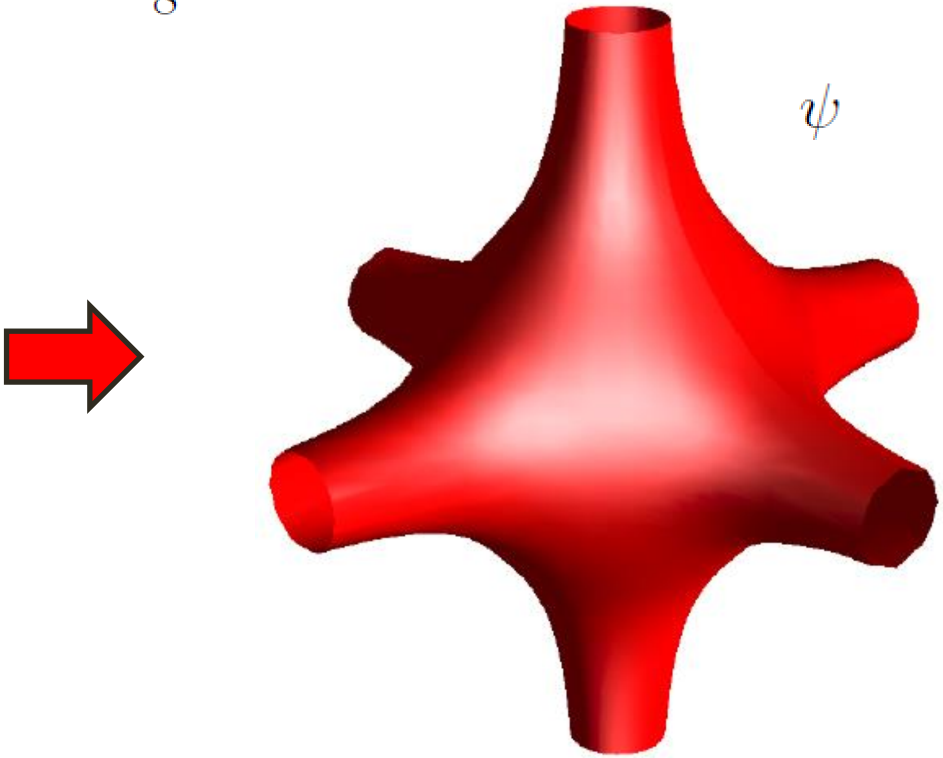


Geometrostatics, on a 3-sphere

Imposing time-symmetry about an initial hyper-surface now gives:

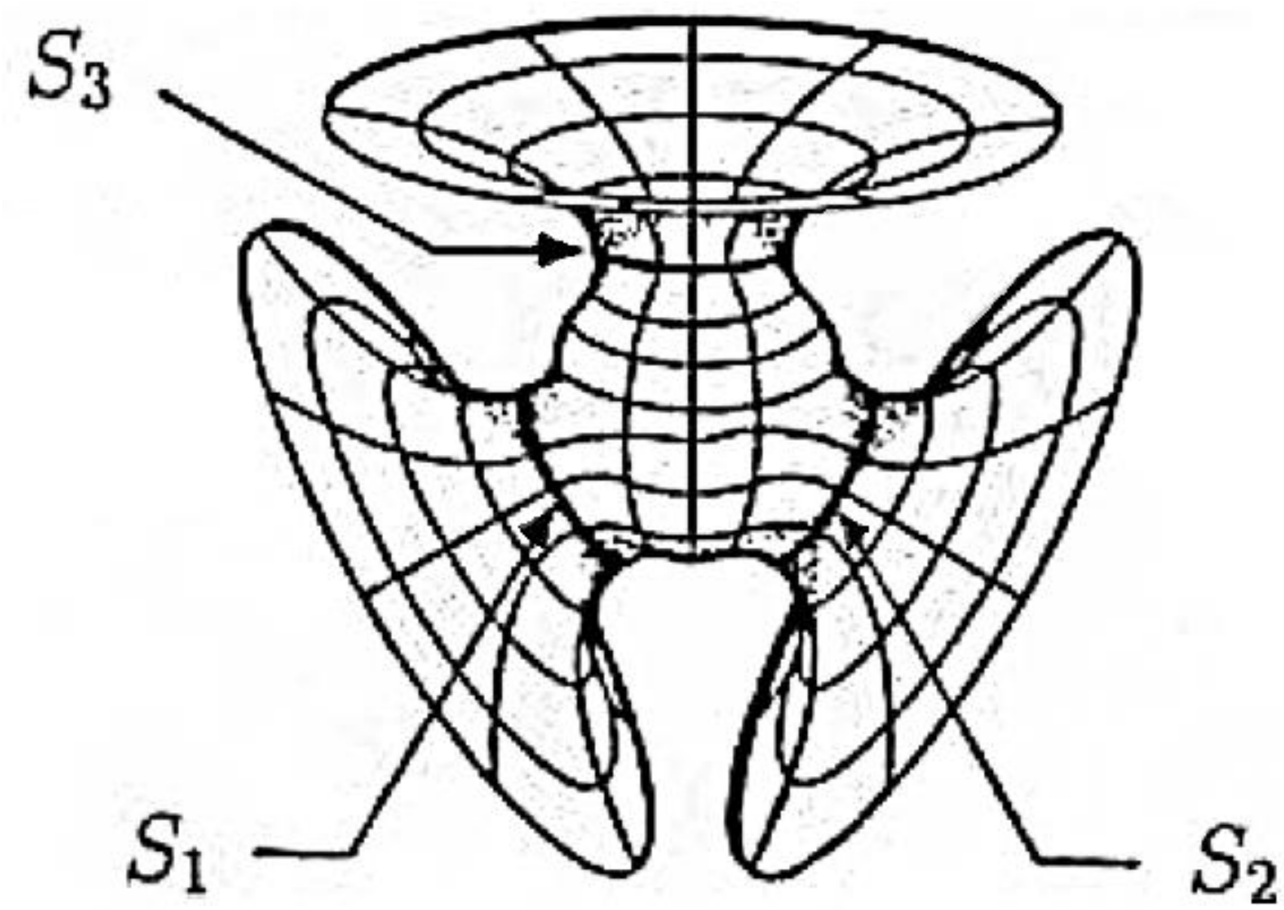
$$dl^2 = \psi^4(d\chi^2 + \sin^2\chi\, d\Omega^2)$$

where $\hat{\nabla}^2\psi = \frac{1}{8}\hat{\mathcal{R}}\psi$



Geometrostatics, on a 3-sphere

Embedding diagram now looks like:



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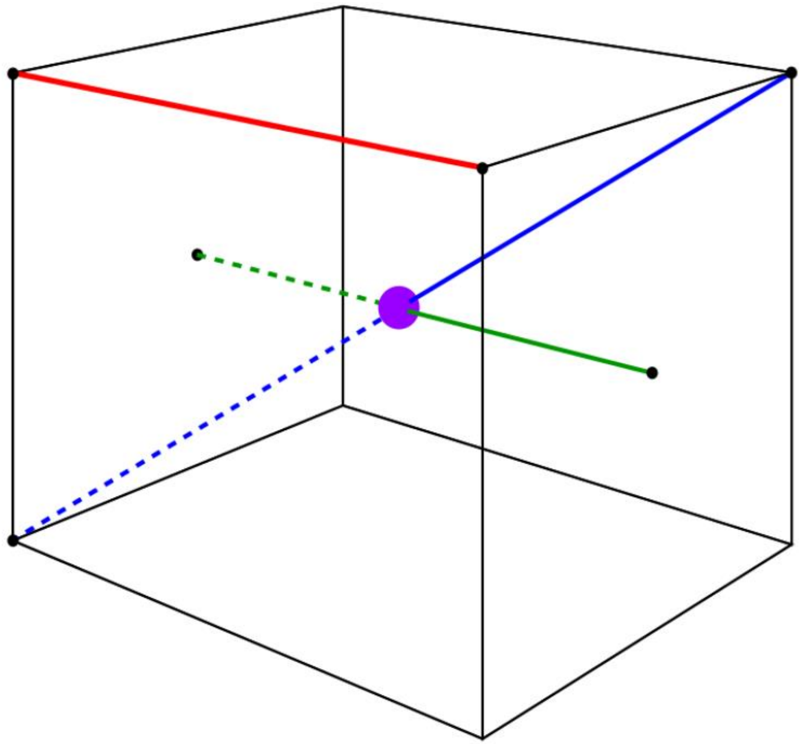
Choosing curves in a lattice cell

Consider an individual cell:



Red curve is the length of a cell edge.

Green curve is the distance from the centre to the middle of a cell face.

Blue curve is the distance from the centre to a corner.

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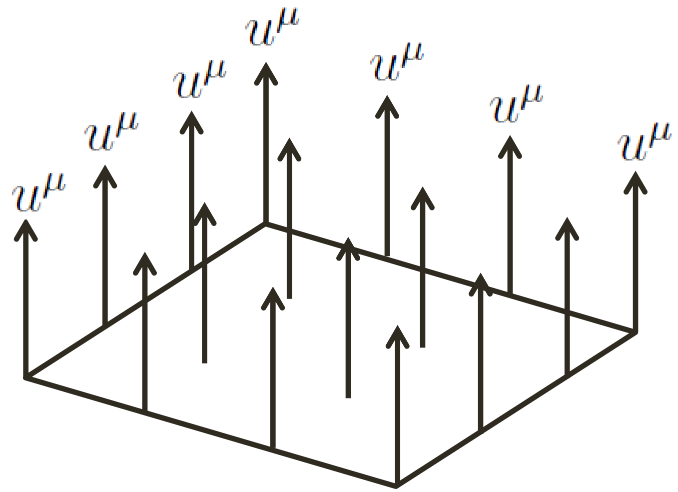
1+3 Decomposition

Take a congruence of time-like curves:



Perform irreducible decompose to get kinematic quantities:

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \theta_{\mu\nu} = -u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - \omega_{\mu\nu},$$

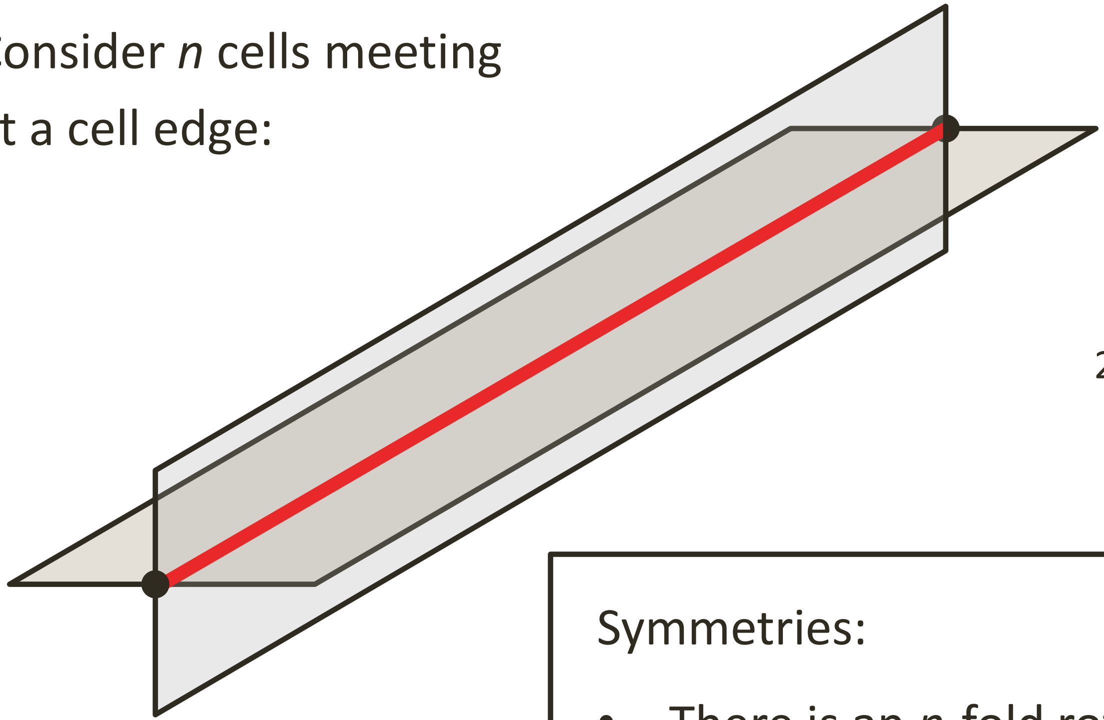
where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and $\dot{X} = u^\mu \nabla_\mu X$.

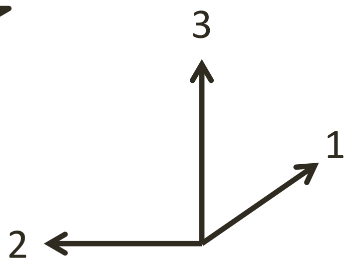
In vacuo, the non-zero parts of the curvature tensor can then be written:

$$E_{\mu\nu} = C_{\mu\rho\nu\sigma} u^\rho u^\sigma$$
$$H_{\mu\nu} = {}^\star C_{\mu\rho\nu\sigma} u^\rho u^\sigma$$

Symmetries around subspaces

Consider n cells meeting at a cell edge:





$$T_2 = T_3 = 0$$

$$T_{23} = 0$$
$$T_{22} = T_{33}$$

Symmetries:

- There is an n-fold rotational symmetry (*implies local rotational symmetry*)
- The are n, or $n/2$, reflective symmetries (*rotational symmetry then implies no preferred orientation*)

Restrictions on geometric quantities

These symmetries imply that the only independent, non-zero geometric quantities on these curves are

$$\Theta, \ \sigma_+, \ E_+ \quad \text{and} \quad H_+.$$

The Ricci and Bianchi identities then give

$$H_+ = 0$$

and

$$\begin{aligned}
\mathbf{e}_0(\Theta) &= -\frac{1}{3}\Theta^2 - \frac{2}{3}(\sigma_+)^2 \\
\mathbf{e}_0(\sigma_+) &= -\frac{1}{3}(2\Theta - \sigma_+)\sigma_+ - E_+ \\
\mathbf{e}_0(E_+) &= -(\Theta + \sigma_+)E_+,
\end{aligned}$$

which are the equations for LRS class II spacetimes.

Cosmological equations

First define:

$$\frac{1}{3}(\Theta - 2\sigma_+) = \frac{\dot{a}_{||}}{a_{||}},$$

$$\frac{1}{3}(\Theta + \sigma_+) = \frac{\dot{a}_{\perp}}{a_{\perp}} .$$

The evolution equations then become:

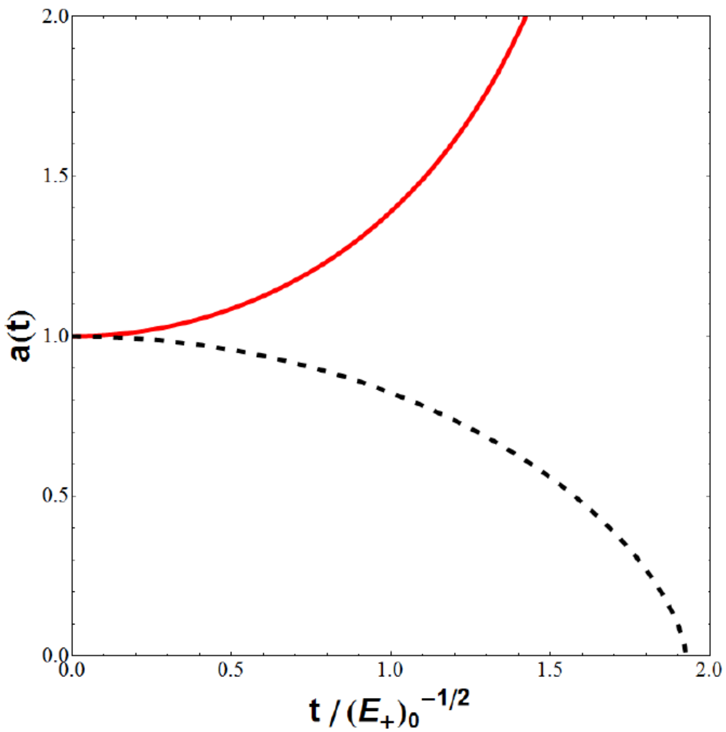
$$\frac{\ddot{a}_{||}}{a_{||}} = \frac{2}{3}E_+ \ , \qquad \frac{\ddot{a}_{\perp}}{a_{\perp}} = -\frac{1}{3}E_+ \ , \qquad \dot{E}_+ + 3\frac{\dot{a}_{\perp}}{a_{\perp}}E_+ = 0 \ .$$

Which give the Friedmann constraint, and proper length of an edge as:

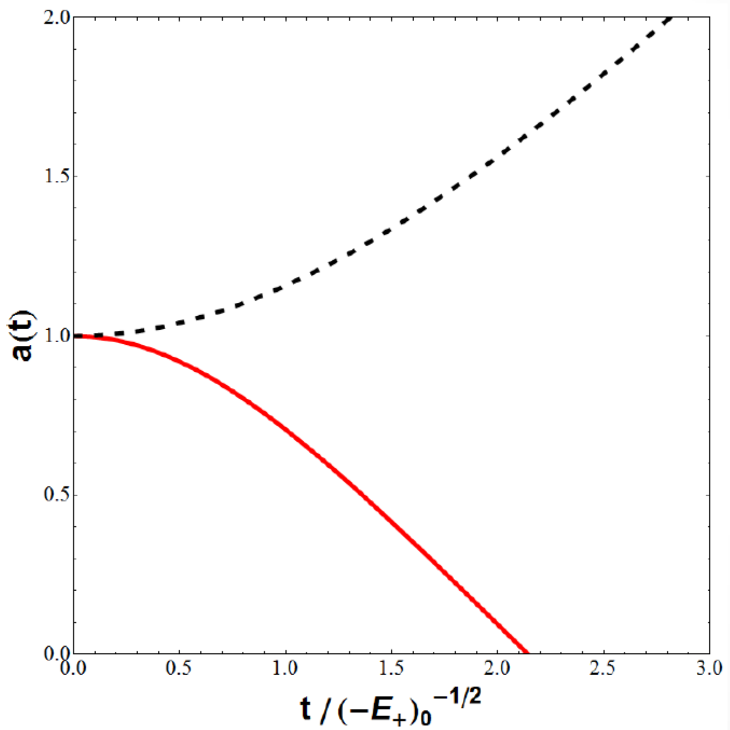
$$\frac{\dot{a}_{\perp}^2}{a_{\perp}^2} = \frac{\frac{2}{3}(E_+)_0}{a_{\perp}^3} - \frac{k}{a_{\perp}^2} \ , \qquad \ell(t) = \int_{\chi_1}^{\chi_2} a_{||}\sqrt{(g_{\chi\chi})_0}\, d\chi \ .$$

Evolution of an element of an edge

$(E_+)_0 > 0$



$(E_+)_0 < 0$



Red curves are for the scale factor along an edge.

Black-dashed curves are for the scale factor perpendicular to an edge.

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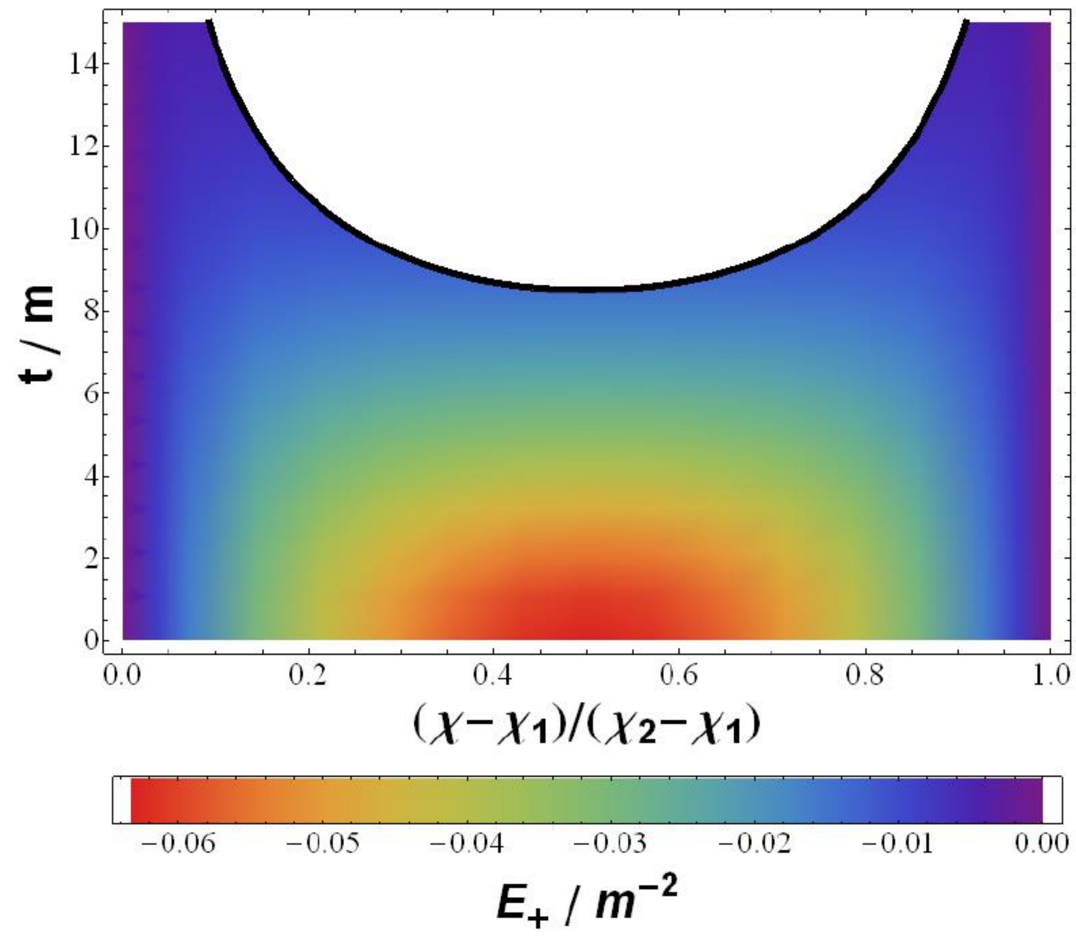
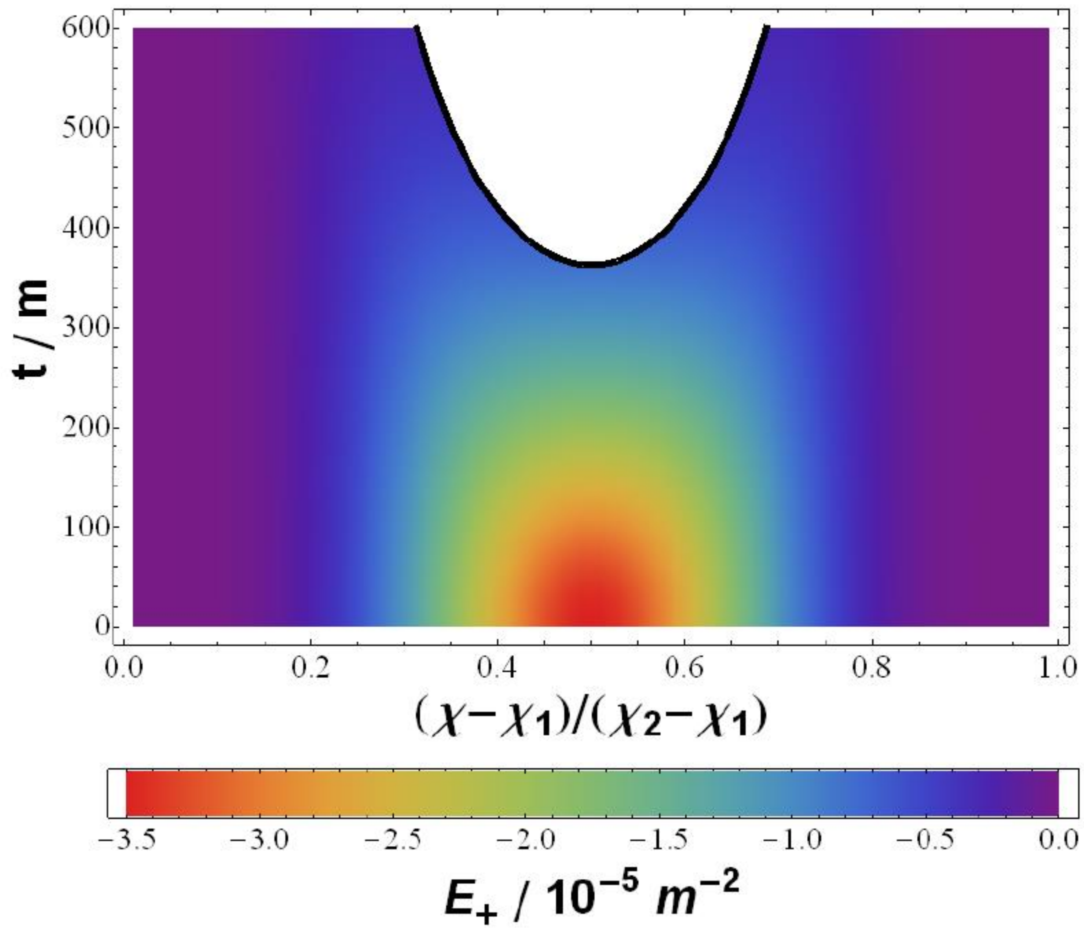
Effective energy density, along an edge

For the 8-black hole universe:

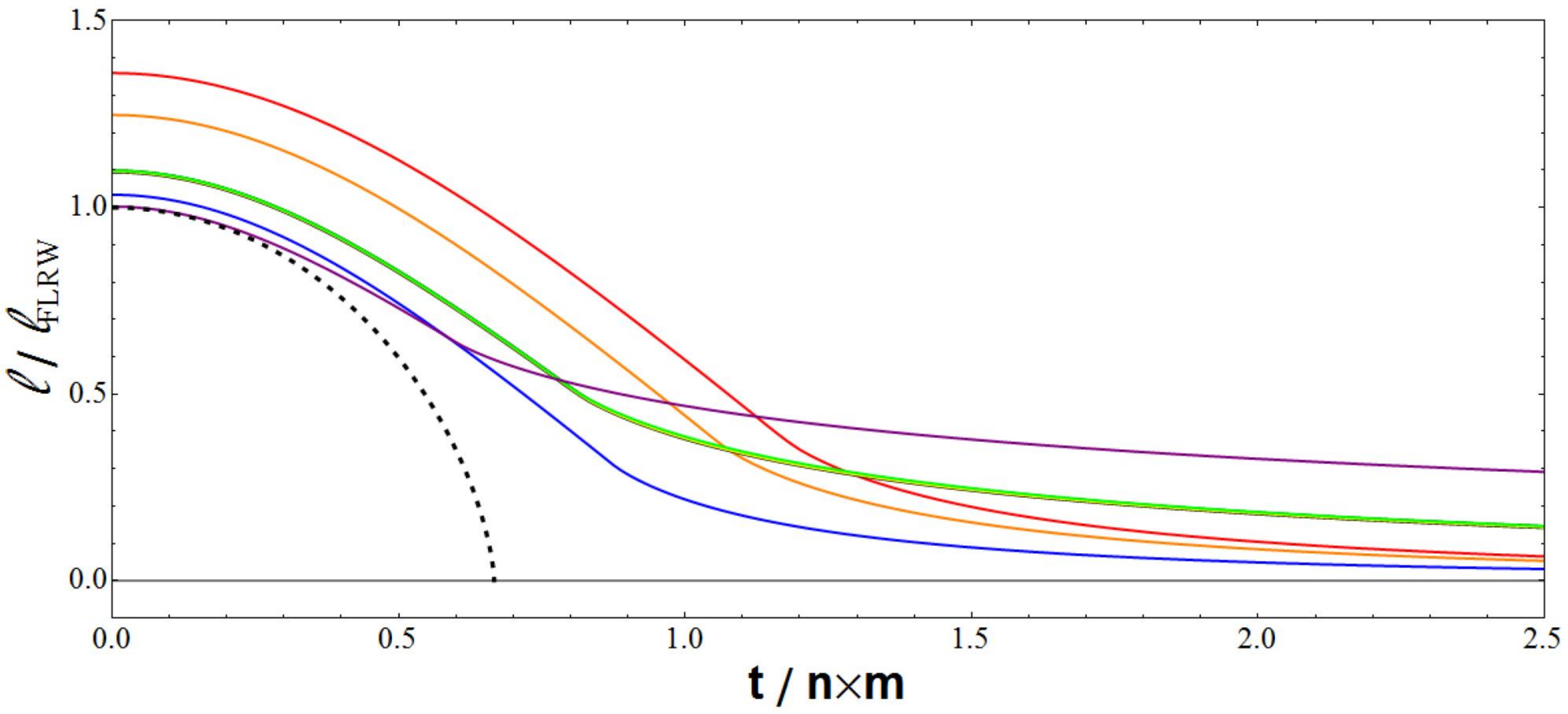


Effective energy density, along an edge

For the 600-black hole universe:



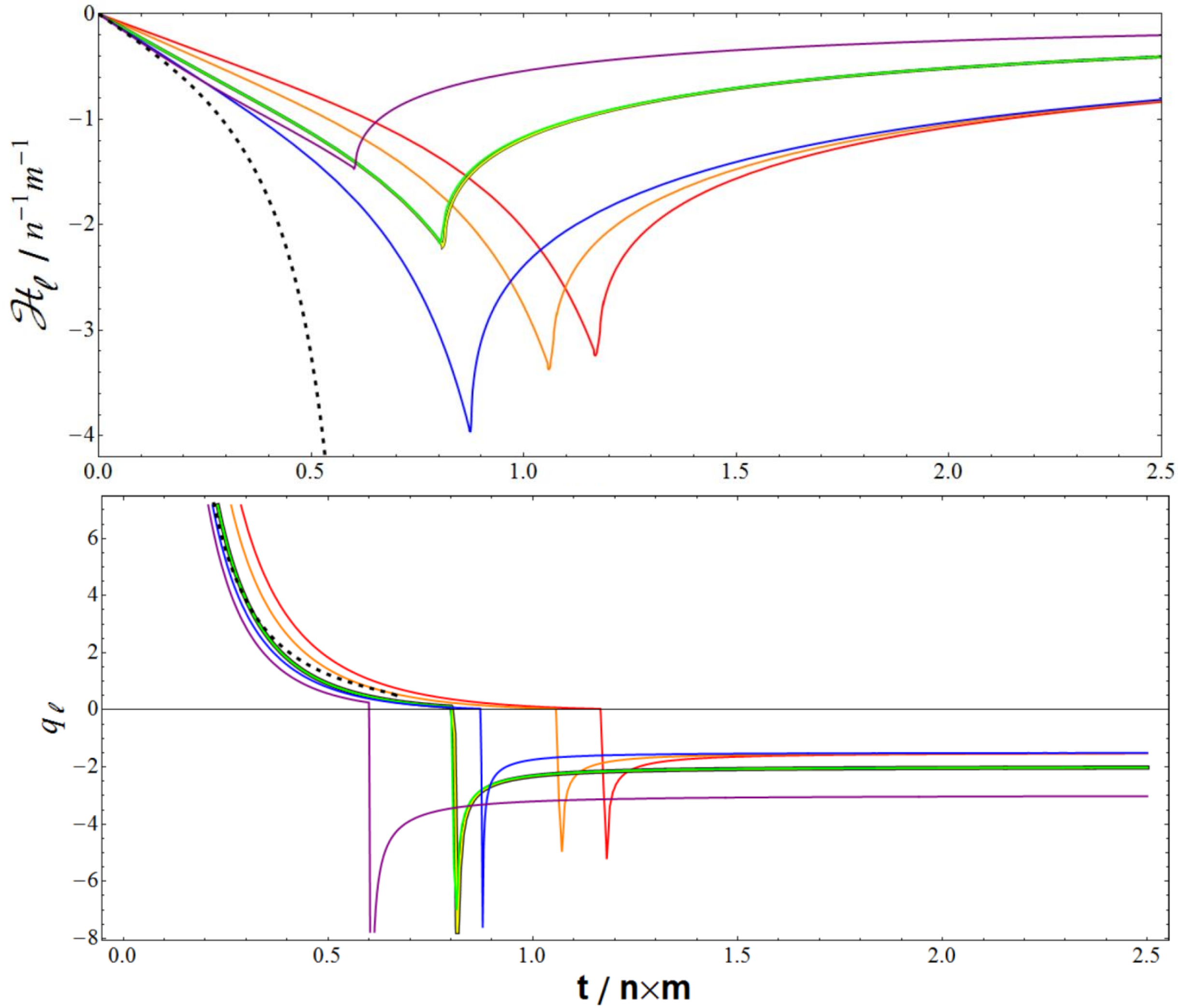
Evolution of a cell edge



Key: 5, 8, 16, 24, 120, and 600 black holes

(*black-dashed line is FLRW*)

Hubble rate, and deceleration parameter



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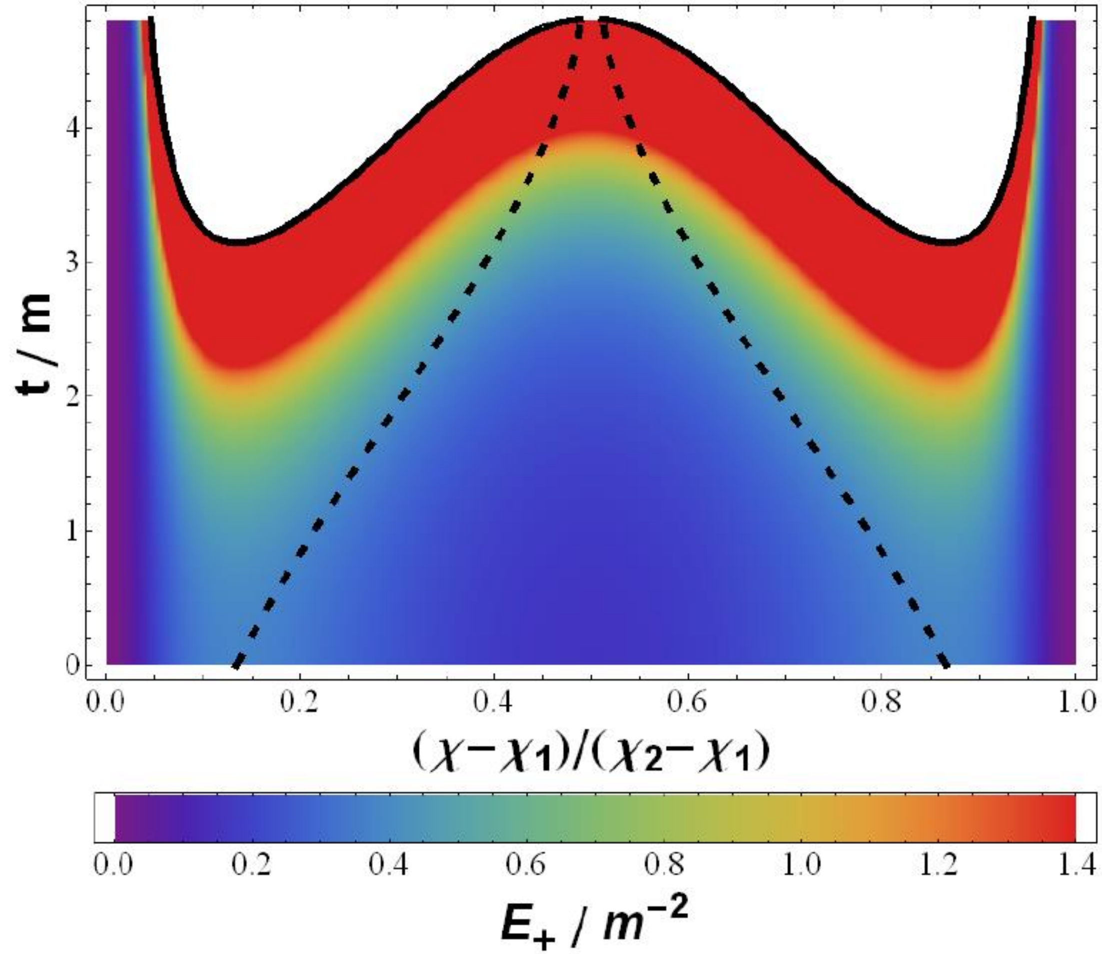
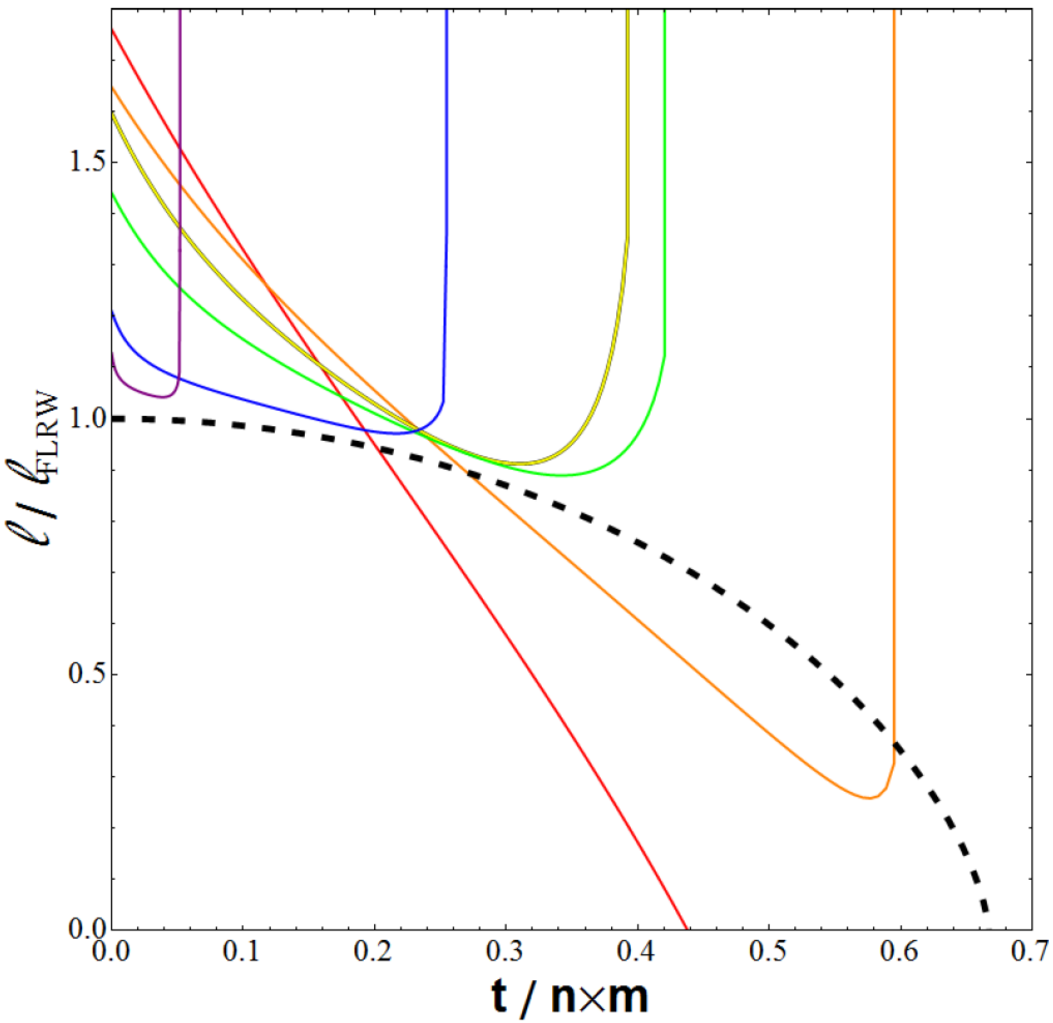
(*black-dashed line is FLRW*)

Effective energy density, through a cell face

For the 8-black hole universe:



Evolution of curve through a cell face



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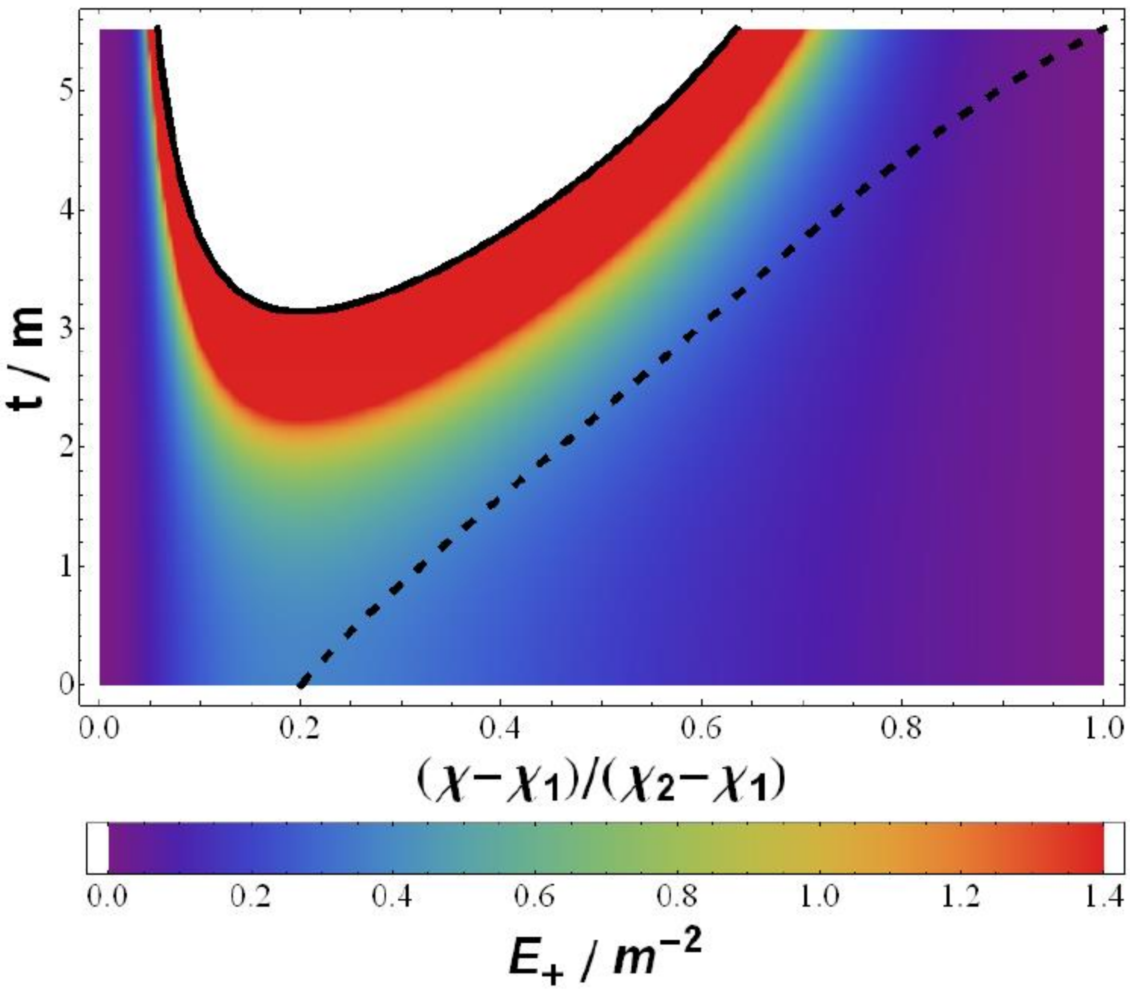
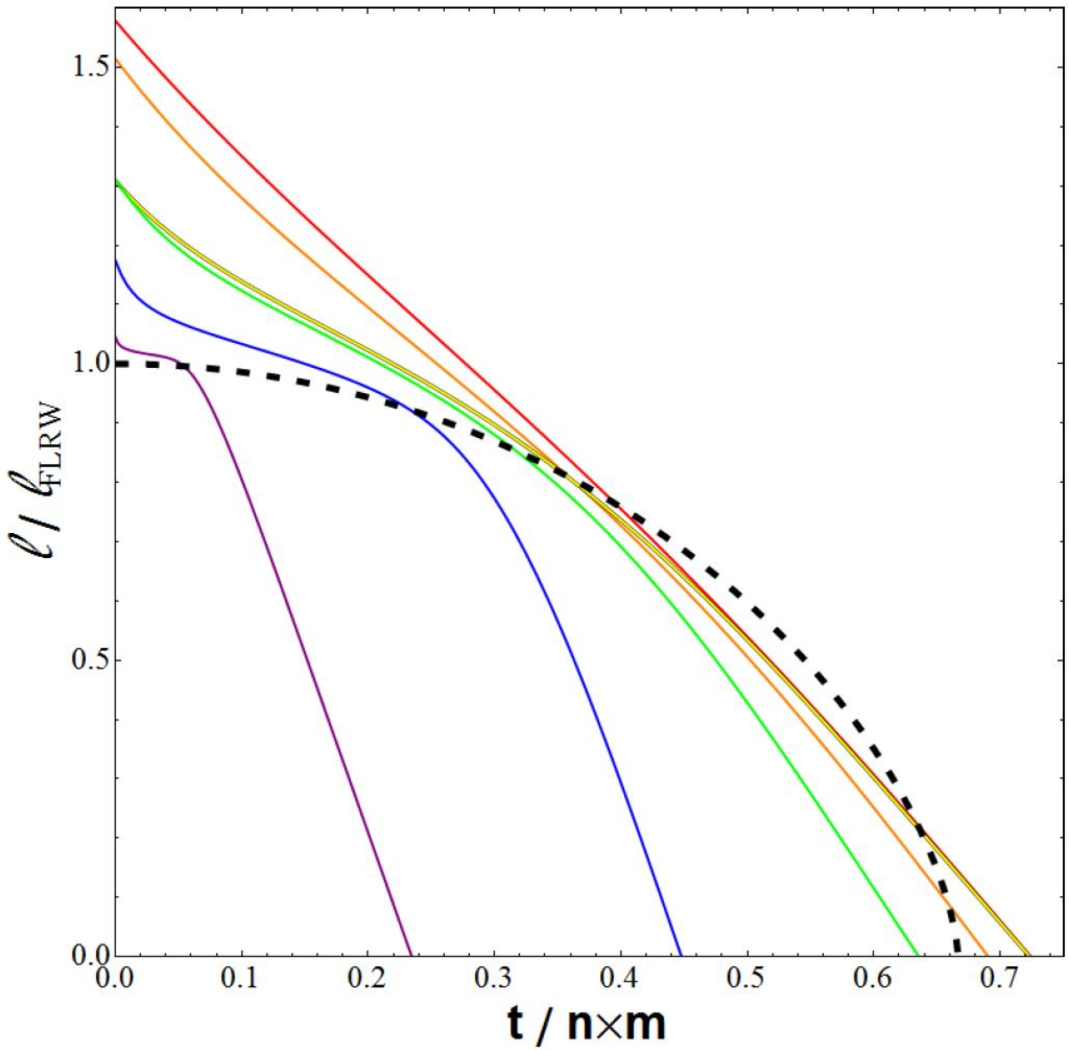
(*black-dashed line is FLRW*)

Effective energy density, through a corner

For the 8-black hole universe:



Evolution of curve through a corner



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Summary

- We have investigated the properties of closed universes that contain a finite number of regularly spaced black holes.
- The scale of these universes, at their maximum of expansion, is a function of the number of black holes present.
- The evolution of the edges of cells evolve away from the maximum of expansion in a similar way to a k=1, dust-filled FLRW universe.
- Away from the maximum of expansion the cell edges are accelerating, and remain non-singular forever.
- The curves that connect black holes collapse to anisotropic singularities at all points in space (except at cell edges).
- The horizon separation distance can either diverge (if the horizon goes singular), or vanish (is the horizon reaches a region of low curvature).
- Not all behaviour converges towards that of a FLRW model as the number of black holes is increased.