Covariant Perturbations of LTB Spacetimes

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Standard Model of Cosmology

Model Building

Constructing a Universe model:

- Gravitation $\rightarrow$ General Relativity
- Matter Content $\rightarrow$ Energy-Momentum Tensor
- Symmetries and Global Structure

Maximally symmetric Universe . . .

- Isotropy $\rightarrow$ The geometry is not dependent on direction!
- Homogeneity $\rightarrow$ The local geometry is the same at all points!

Isotropy at all points $\rightarrow$ Homogeneity!

Homogeneity at all points $\neq$ Isotropy!

Homogeneity and isotropy $\rightarrow$ Friedmann-Lemaître-Robertson-Walker (FLRW)

Standard Model of Cosmology

Homogeneity and Isotropy

Copernican Principle

We are not at a special location in the Universe!

Cosmological Principle

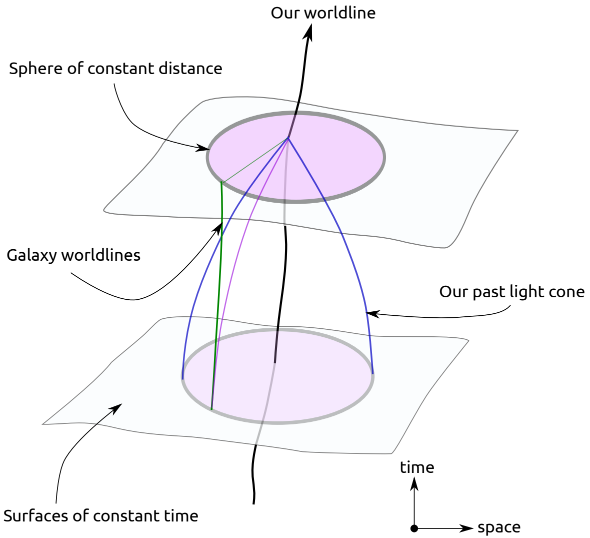
Smoothed on large enough scales, the Universe is spatially homogeneous and isotropic!

Isotropy is well constrained by the CMB . . .

Homogeneity is **not** directly observable, it is infered!

Standard Model of Cosmology

Isotropy and Homogeneity



Standard Model of Cosmology

Isotropy and Homogeneity

- Testing Copernican principle is hard → We view the universe from one spacetime event!

Exact Statement . . .

"If all observers measure their observables to be exactly isotropic, then the spacetime is exactly FLRW."

Realistic Statement . . .

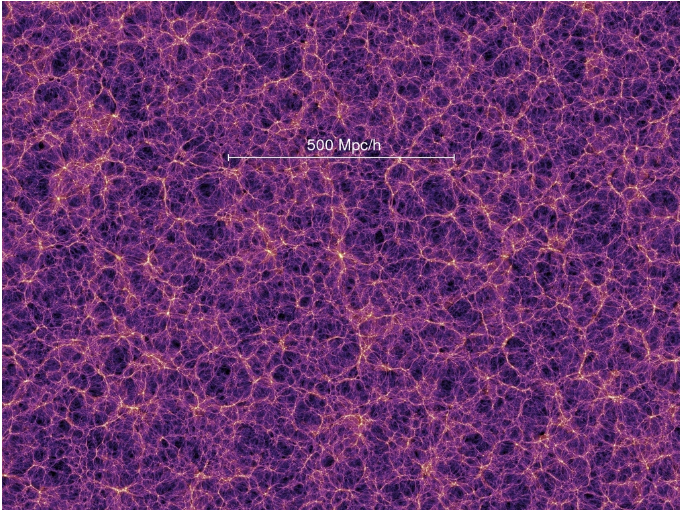
"If most observers find their observables consistent with a small level of anisotropy, then the metric of the Universe is roughly FLRW when smoothed over a suitably large scale."

- Smoothing in GR?
- Spatial gradients?
- Can only test Copernican principle locally (sub Gpc) . . .
- Isotropic and homogeneous → Highly symmetric . . . the real Universe is not!

Standard Model of Cosmology

Isotropy and Homogeneity

Real Universe has structure that breaks homogeneity and isotropy $\rightarrow$ Need perturbation theory!



Either way, let's recount the FLRW models . . .

- Isotropy and homogeneity → FLRW
- Metric:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + f_K^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \quad (1)$$

- Assume perfect fluid:

$$T_{ab} = \mu u_a u_b + p h_{ab} \quad (2)$$

- Einstein's Field Equations:

$$H^2 = \frac{\kappa}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{r} \quad (3)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\mu + 3p) + \frac{\Lambda}{3} \quad (4)$$

- Matter conservation:

$$\dot{\mu} = -3H(\mu + p) \quad (5)$$

Inhomogeneity

Motivation

- Relax global homogeneity $\rightarrow$ First step to an inhomogeneous background
- Isotropy constraints $\rightarrow$ Spherical symmetry!
- Assume radial inhomogeneity $\rightarrow$ Energy-Momentum content is pure dust
- Open Problems:
  - Structure Formation
  - Model Testing
  - Understanding Observations

- Spherically symmetric, radially inhomogeneous.

Metric of Lemaitre-Tolman-Bondi (LTB) Spacetime

$$ds^2 = -dt^2 + X^2(t,r)dr^2 + A^2(t,r)d\Omega^2 \tag{6}$$

$$X(t,r) = f(r)a_{\parallel}(t,r)) \qquad f(r) = \left[1 - \kappa(r)r^2\right]^{-1/2} \tag{7}$$

- Setting:

$$ra_{\perp}(t,r) = A(t,r) \qquad a_{\parallel}(t,r) = \partial_r A(t,r)$$

- Anisotropic expansion:

$$H_{\parallel}(t,r) = \frac{\dot{a}_{\parallel}(t,r)}{a_{\parallel}(t,r)} \;\; ; \;\; H_{\perp}(t,r) = \frac{\dot{a}_{\perp}(t,r)}{a_{\perp}(t,r)}$$

A Tale of Two Formalsims...

We will seek to relate two different approaches to General Relativity:

Approach 1

2+2 Covariant Formalism:

- Originally used for stellar/black hole perturbations
- Gauge-invariant framework in place... not geometrically clear!

Approach 2

1+1+2 Covariant Formalism

- Based on 1+3 formalism
- Originally used for cosmological perturbation theory
- Physically and geometrically meaningful... gauge-invariance natural!

2+2 Formalism

(This may be a little heavy...)

- Decompose full spacetime:

$$\mathcal{M} = \mathcal{M}^2 \times S^2 \tag{8}$$

- Re-write LTB metric in 2+2 form[1]:

$$ds^2 = -dt^2 + X^2(t,r)dr^2 + A^2(t,r)d\Omega^2$$

$$ds^2 = g_{AB}dx^A dx^B + r^2\gamma_{ab}dx^a dx^b$$

- Energy-Momentum tensor:

$$t_{\mu\nu} = \mathrm{diag}(\mathrm{t_{AB}(x^C), Q(x^C) r^2 \gamma_{ab}}) \tag{9}$$

- Einstein's equations:

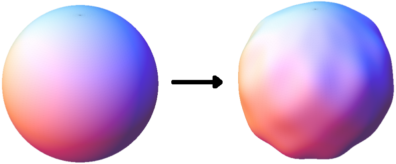
$$G_{AB} = 8\pi t_{AB} \tag{10}$$

$$G^a{}_a = 16\pi Q \tag{11}$$

(12)

[1] $A, B, \ldots \in \{0, 1\}$ and $a, b, \ldots \in \{2, 3\}$

- Introduce linear perturbations: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$



- Spherical symmetry $\rightarrow$ tensor harmonic decomposition!

- Perturbations decompose into polar (even) and axial (odd) perturbations

$$h_{\mu\nu} = \sum_{\ell m} \left[(h^{\ell m}_{\mu\nu})^{(P)} + (h^{\ell m}_{\mu\nu})^{(A)} \right] \quad (13)$$

2+2 Formalism

Metric Perturbations II

- Expand perturbed metric into multipoles:

$$h_{\mu\nu}^{\text{Polar}} = \begin{pmatrix} h_{AB} Y & h_A Y_{:a} \\ \text{Symm.} & r^2 (K Y \gamma_{ab} + G Y_{:ab}) \end{pmatrix}$$

$$h_{\mu\nu}^{\text{Axial}} = \begin{pmatrix} 0 & \bar{h}_A \bar{Y}_a \\ \text{Symm.} & h \, \bar{Y}_{ab} \end{pmatrix} \tag{14}$$

- Do similar for energy-momentum tensor ...

- Spherical symmetry $\rightarrow$ Axial and Polar dynamically independent

- Want gauge invariant linear perturbations:

$$\mathbf{X} \to \mathbf{X} + \delta\mathbf{X} \tag{15}$$

- Infinitesimal coordinate transformation:

$$x^{\mu} \to x^{\mu'} = x^{\mu} + \xi^{\mu}; \quad \xi^{\mu} \ll 1 \tag{16}$$

- Yields a new tensor field for the perturbation!

$$\delta\mathbf{X} \to \delta\mathbf{X}' = \delta\mathbf{X} + \mathcal{L}_{\xi}\mathbf{X} \tag{17}$$

- Gauge invariant iff $\mathcal{L}_{\xi}\mathbf{X} = 0$

- Gauge invariants are physical!

Gauge Invariant Perturbations

Gauge-invariant metric perturbations $\leftrightarrow$ Symmetries of background spacetime

How do we get our gauge invariant (GI) variables?

- Gauge freedom: $h_{\mu\nu} \sim h_{\mu\nu} + (\mathcal{L}_\xi g)_{\mu\nu}$

What do we do . . .

- Construct gauge-invariant combinations of $h_{\mu\nu} \rightarrow$ GMG[2] ✓
- Adopt the Regge-Wheeler (RW) gauge: $h = h_A^{\text{Polar}} = G = 0 \rightarrow \checkmark$

- Can transform to any other gauge . . . RW is convenient!

[2] Gundlach and Martín-García (2000)

Let's start off in the polar sector!

- Background unit vectors in M^2:

$$\hat{u}^A = (1,0) \;\; ; \;\; \hat{n}^A = (0, X^{-1}) \;\; ; \;\; \epsilon_{AB} u^B = -n_A$$

- In RW gauge $h_{AB} \rightarrow k_{AB}$
- Decompose into frame $\{u^A, n^A\}$

Metric Perturbations

$$k_{AB} = \eta \left(-u_A u_B + n_A n_B\right) + \phi \left(u_A u_B + n_A n_B\right) + \varsigma \left(u_A n_B + n_A u_B\right) \tag{18}$$

Convenient Variables

$$\chi = \phi - \varphi + \eta \tag{19}$$

- EFE tell us that $\eta \rightarrow 0$ for $\ell \geq 2$!
- Substitute into equations, project onto $\{u, n\}$ basis $\rightarrow$ Scalar equations!

2+2 Formalism

Gauge-Invariants

- Structure of polar perturbations to the LTB metric[3]:

$$ds^2 = -\left[1 - (\chi + \varphi)Y\right] dt^2 - 2\,\varsigma\, X(t,r) Y\, dt dr$$
$$\left[1 + (\chi + \varphi)Y\right] X^2(t,r)\, dr^2 + \left[1 + \varphi Y\right] A^2(t,r) d\Omega^2$$

Parameterise 4-velocity as per GMG[4]

$$u_\mu = \left[\hat{u}_A + \left(w\hat{n}_A + \frac{1}{2} h_{AB}\hat{u}^B\right) Y, vY_a\right]$$

$$\rho = \rho^{\mathrm{LTB}}\left(1 + \Delta Y\right)$$

[3] Clarkson, Clifton and February, (2009), JCAP, 06, 025
[4] Gundlach and Martín-García, PRD, 61, 084024

Polar Sector:

- Master equations for $\ell > 1 (\eta = 0)$ have been derived[5]

Evolution Equations (Not Full System)

$$
\begin{aligned}
-\ddot{\chi} + \chi'' + (\dots)\,\varsigma' &= S_\chi \\
-\ddot{\varphi} + (\dots)\,\varsigma' &= S_\varphi \\
\dot{\varsigma} &= S_\varsigma
\end{aligned}
$$

Constraint Equations

$$
\begin{aligned}
8\pi\rho w &= \dot{\varphi}' + C_w \\
8\pi\rho\Delta &= -\varphi'' + (\dots)\,\varsigma' + C_\Delta \\
16\pi\rho v &= \varsigma' + C_v
\end{aligned}
$$

- Frame derivatives: $\dot{f} = u^A f_{|A} \quad ; \quad f' = n^A f_{|A}$
- Free Cauchy data: $u_{\rm Free} = \{\chi, \varphi, \varsigma, \dot{\chi}, \dot{\varphi}\}$.

[5] Clarkson, Clifton and February, (2009), JCAP, 06, 025

Axial Metric:

$$ds^2 = -dt^2 + X^2(t,r)dr^2 + A^2 d\Omega^2 + 2k_A \bar{Y}_b dx^A dx^b \tag{20}$$

$$u_\mu = (\hat{u}_A, \bar{v}\bar{Y}_a) \tag{21}$$

Axial Sector:

$$\Pi = \epsilon^{AB}\left(\frac{k_A}{r^2}\right)_{|B} \quad \leftrightarrow \quad \text{Master Variable} \tag{22}$$

$$\left[\frac{1}{r^2}\left(r^4\Pi\right)_{|A}\right]^{|A} - (\ell-1)(\ell+2)\Pi = -16\pi\epsilon^{AB}L_{A|B} \tag{23}$$

$$\dot{\bar{v}} - c_s^2\left(2U+\mu\right)\dot{\bar{v}} = 0 \quad \leftarrow \quad c_s^2 = 0 \text{ in LTB} \tag{24}$$

Free Cauchy data: $u_{\text{Free}} = \{\bar{v}, \Pi, \dot{\Pi}\}$

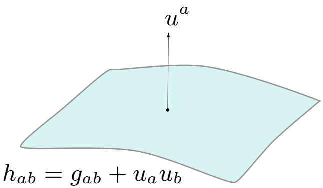
1+1+2 Formalism

(Now for something different...)

1+3 Formalism

Introduction

- Partial tetrad formalism[6] (Ehlers, Ellis, Hawking, ...)
- Project orthogonally to timelike congruence u^a



Einstein's Field Equations $\leftrightarrow$

- Ricci Identities:

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d \leftrightarrow \text{Kinematic Evolution}$$

- Twice Contracted Bianchi Identites:

$$\nabla_b T^{ab} = 0 \leftrightarrow \text{Conservation Equations}$$

- Bianchi Identities:

$$\nabla_{[a}R_{bc]de} = 0 \leftrightarrow \text{Evolution of Weyl Tensor}$$

[6] Note that indices here are defined as $a, b, \ldots \in \{0, 1, 2, 3\}$

- Derivatives:

$$\dot{f} = u^a \nabla_a f$$

$$D_a f = h_a{}^b \nabla_b f$$

- Kinematics:

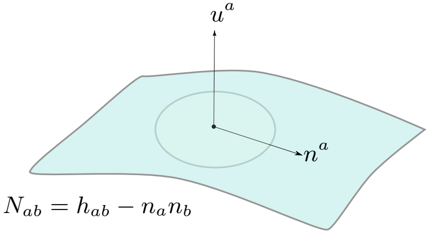
$$\nabla_a u_b = \overbrace{-u_a \dot{u}_b}^{\text{Acceleration}} + \overbrace{\frac{1}{3}\Theta h_{ab}}^{\text{Expansion}} + \overbrace{\sigma_{ab}}^{\text{Shear}} + \overbrace{\omega_{ab}}^{\text{Vorticity}}$$

- Energy-Momentum:

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab}$$

- 1+3 $\rightarrow$ System of ODES involving 1+3 scalar quantities
- Inhomogeneity $\rightarrow$ Breaks simple structure
- Non-zero vectors and tensors $\rightarrow$ coupling terms!
- Recover simple structure by another vector field ...

- Adopt formalism of Clarkson and Barrett[7]
- Spacelike congruence $n^a \rightarrow$ *further* split of $1 + 3$ equations
  - Projection tensor onto 2-sheets orthogonal to u^a and n^a:



- Spacetime objects split into: Scalars , 2-Vectors , Transverse-Traceless 2-Tensors.
- Supplement 1+3 equations with Ricci identity for n^a:
  $R_{abc} \equiv 2\nabla_{[a}\nabla_{b]}n_c - R_{abcd}e^d = 0.$

[7] Clarkson and Barret, (2003), CQG, 20, 3855

- Example: Decomposition of 3-Vectors:

$$\psi_a = \mathbf{\Psi} n^a + \mathbf{\Psi}^a$$

- Term parallel to n^a
- Term lying in sheet orthogonal to u^a and n^a

- Example: Decomposition of 3-tensors:

$$\psi_{ab} = \mathbf{\Psi}\left(n_a n_b - \frac{1}{2} N_{ab}\right) + 2\mathbf{\Psi}_{(a} n_{b)} + \mathbf{\Psi}_{ab}. \qquad (25)$$

- New Derivatives:

$$\hat{\psi}_{a\dots b} = n^e D_e \psi_{a\dots b}$$

$$\delta_e \psi_{a\dots b} = N_e{}^j N_a{}^f \dots N_b{}^g D_j \psi_{f\dots g}$$

A Useful Dictionary...

Θ	Expansion of u^a
a^a	Sheet Acceleration
ϕ	Sheet Expansion
ξ	Rotation of $n^a \rightarrow$ Twisting of Sheet
ζ_{ab}	Shear of $n^a \rightarrow$ Distortion of Sheet
$\mathcal{A}$	Radial component of acceleration of u^a
$\mathcal{A}^a$	Acceleration of u^a lying in the sheet orthogonal to n^a
α^a	Acceleration of n^a
$\{\mathcal{E}, \mathcal{E}_a, \mathcal{E}_{ab}\}$	Projections of Electric Weyl Tensor E_{ab}
$\{\mathcal{H}, \mathcal{H}_a, \mathcal{H}_{ab}\}$	Projections of Magnetic Weyl Tensor H_{ab}
μ	Energy Density
$\{\Sigma, \Sigma_a, \Sigma_{ab}\}$	Projections of Shear Tensor σ_{ab}
$\{\Pi, \Pi_a, \Pi_{ab}\}$	Projections of Anisotropic Pressure ω^a
$\{\Omega, \Omega_a\}$	Projections of Vorticity Vector ω^a
$\{Q, Q_a\}$	Projections of Heat Flux q^a

A 1+1+2 Approach[8]:

- LTB Spacetime characterised covariantly by:

$$\mathbf{X}_{\rm LTB} = \{\mathcal{E}, \phi, \mu, \Theta, \Sigma, \hat{\mu}, \hat{\Theta}\} \rightarrow \mathcal{O}[0] \tag{26}$$

Background Equations

$$\dot{\mathbf{X}}_{\rm LTB} = \dots \tag{27}$$

$$\hat{\mathbf{X}}_{\rm LTB} = \dots \tag{28}$$

$$\mathcal{C}_{\mathbf{X}_{\rm LTB}} = \dots \tag{29}$$

- Linear perturbations → Full 1+1+2 equations → Kill terms $\mathcal{O}[2]$ and higher ...
- Set of gauge-invariant variables, Υ:

$$\Upsilon_a = \delta_a \mathbf{X}_{\rm LTB} \tag{30}$$

- Gauge invariance guaranteed by the Stewart-Walker Lemma!

[8] Pratten and Clarkson, In Prep.

Linearised equations in LTB[9]:

- Harmonic decomposition

Scalar, Vector and Tensor Harmonics

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_S\, Q \quad (31)$$

$$\boldsymbol{\Psi}_a = \boldsymbol{\Psi}_V\, Q_a + \bar{\boldsymbol{\Psi}}_V\, \bar{Q}_a \quad (32)$$

$$\boldsymbol{\Psi}_{ab} = \boldsymbol{\Psi}_T\, Q_{ab} + \bar{\boldsymbol{\Psi}}_T\, \bar{Q}_{ab} \quad (33)$$

- Perturbation variables:

$$\boldsymbol{\Phi} = \{\mathcal{H}_{ab}, \mathcal{E}_{ab}, \zeta_{ab}, \Sigma_{ab}, \mathcal{E}_a, \dots\}^{\mathsf{T}}$$

- Linearised equations:

$$\gamma\dot{\boldsymbol{\Phi}} + \lambda\hat{\boldsymbol{\Phi}} = \Gamma\boldsymbol{\Phi}$$

- Master equation:

$$-\ddot{W}_{\{ab\}} + \hat{\hat{W}}_{\{ab\}} + (\dots)\hat{W}_{\{ab\}} + (\dots)\dot{W}_{\{ab\}} + \delta^2 W_{\{ab\}} + (\dots)W_{\{ab\}} = \mathcal{S}_{ab}$$

[9] Pratten and Clarkson, In Prep.

Correspondence

Correspondence between 2+2 and 1+1+2 $\rightarrow$ Best of both worlds![10]

- Ansatz for radial vector:

$$n_\mu = \left[\hat{n}_A + \left(w\hat{u}_A + \frac{1}{2}h_{AB}\hat{n}^B\right)Y, gY_a\right] \tag{34}$$

- GR $\rightarrow$ Freedom to choose frame basis in tangent space!
- g $\rightarrow$ Frame degree of freedom!

Decompose 2+2 perturbations into 1+1+2 variables ...

$$\mathcal{E}_T = -\frac{1}{2}\frac{(\chi + \varphi)}{r^2} \tag{35}$$

$$\bar{\mathcal{H}}_T = -\frac{1}{2}\frac{\varsigma}{r^2} \tag{36}$$

$$\Sigma_T = \frac{v}{r^2} \tag{37}$$

$$\zeta_T = \frac{g}{r^2} \tag{38}$$

$$\ldots$$

[10] Pratten and Clarkson, In Prep.

Axial Sector:[11]

Master Variable

$$W_{ab} = r^2 \delta_{\{a}\delta_{b\}}\mathcal{H} \tag{39}$$
$$W_T = \Pi \tag{40}$$

Master Equation

$$-\ddot{W} + \hat{\hat{W}} + \dot{W}\left(2\Sigma - \frac{7}{3}\Theta\right) + 3\phi\hat{W} - \left[2\mu + \frac{\ell(\ell+1)}{r^2} - \frac{6}{r^2}\right]W = -\frac{2}{r^2}\widehat{[\mu\Omega r^2]} \tag{41}$$

Axial gravitaitonal waves sourced by density and vorticity gradients!

[11] Pratten and Clarkson, In Prep.

Polar Sector (Preliminary)[12]

Master Variables

$$\chi = 2\mathcal{E}_T\left[2K - \frac{\ell(\ell+1)}{r^2}\right] + \frac{3}{2}\Sigma\phi\bar{\mathcal{H}}_T + \frac{\bar{\mathcal{H}}_V}{r}\left[3\Sigma - \frac{4}{3}\Theta\right] + 2\phi\frac{\mathcal{E}_V}{r} - \frac{2}{3}\frac{M_V}{r} + 2\frac{X_V}{r}$$

$$\varphi = -2\mathcal{E}_T - \chi$$

$$\varsigma = -2\mathcal{H}_T$$

- χ is GW degree of freedom $\rightarrow$ couples to matter terms!
- φ encapsulates matter content!
- $\varsigma \rightarrow$ GW degree of freedom and frame dragging!

- System of 2+2 equations numerically solved! $\rightarrow$ February et al, arXiv:1311.5241
- Need tensor to unify polar and axial sectors! $\rightarrow$ Work in progress . . .

[12] Pratten and Clarkson, In Prep.

Conclusions

2+2 Formalism:

- Construct perturbed metric for LTB spacetimes in 2+2 formalism
- Construct 1+1+2 variables in terms of 2+2 variables
- Find fundamental degrees of freedom to invert 1+1+2 variables
- *Now* rewriting master equations for LTB in terms of 1+1+2 variables

1+1+2 Formalism:

- System of propagation, evolution and constraint equations
- Construct gauge-invariant variables
- Construct wave equations
- *Now* isolating master variables

Applications:

- Propagation of gravitational waves in inhomogeneous background
- Structure formation
- Baryon acoustic oscillations, integrated Sachs-Wolfe, ...

Also:

- Correspondence works in Schwarzschild!