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3.5 PN spin-orbit effects in the phasing of inspiralling compact binaries

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Outline

- Motivation and introduction to PN
- Effective pole-dipole formalism: spinning point particles
- Sketch of the computation of the 3.5 PN spin-orbit effects
  - Equations of motion (and associated dynamical quantities)
  - Flux
- Estimates of the phase

Motivation: building accurate templates for gw detection

- Inspiralling binaries of compact objects (black holes and/or neutron stars) are one of the most promising sources of GW that we hope to detect with the advanced versions the ground based detectors LIGO and Virgo and with a future space-based detector.

- Successfully extracting the very weak signal from the noise and estimating the parameters of the source with good precision can be achieved using matched filtering techniques provided that we have a very accurate modelling of the waveform.

- The post-Netwtonian approximation scheme enables to compute such accurate waveforms for the inspiral phase. For non-spinning compact binary systems, such templates are known to 3.5 PN order for the phase (3PN for the amplitude).

- We now have observational evidence pointing towards the existence of fast-rotating black holes (see e.g. *Reynolds* 2013)

⟶ This has motivated a lot of effort to include spin effects to the same level of accuracy

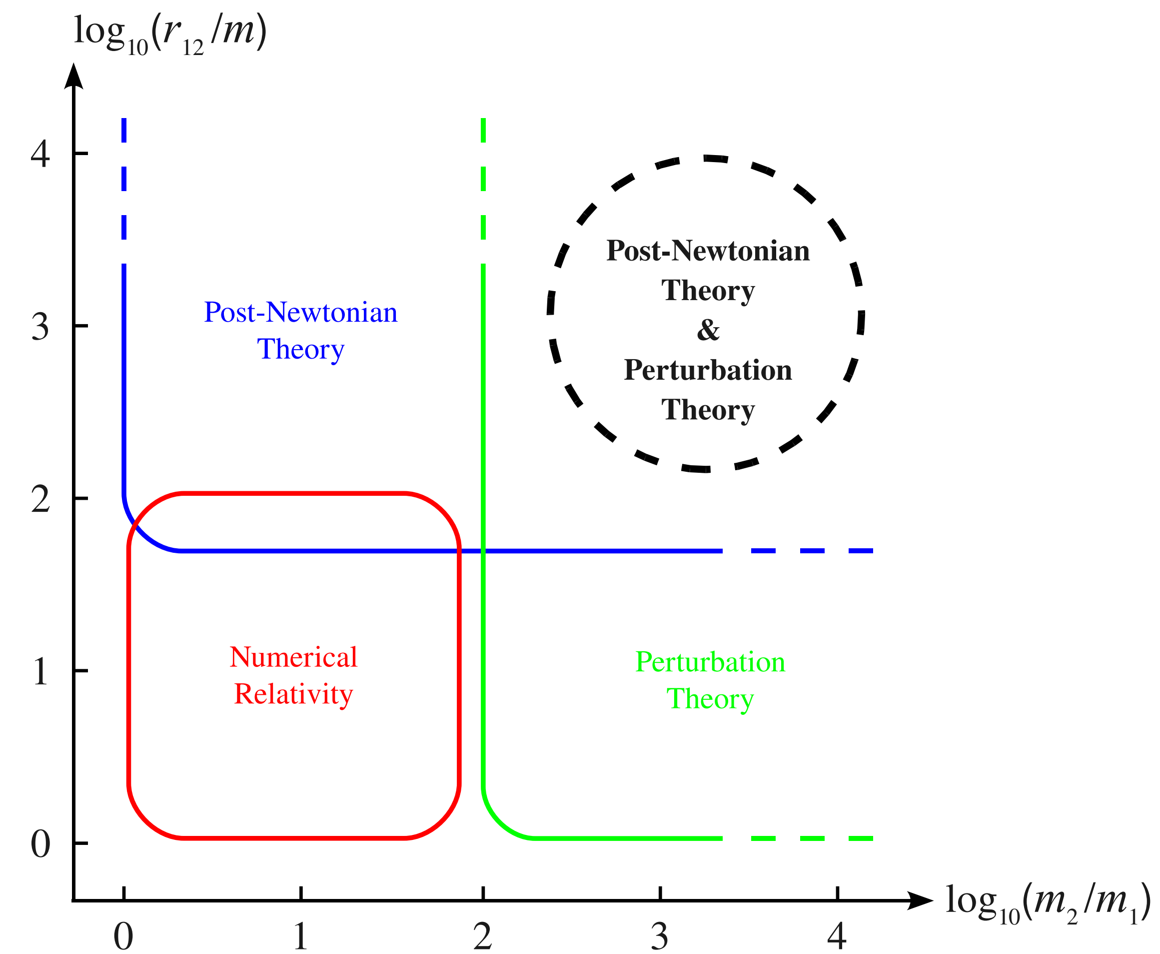
Brief intro to PN

The PN expansion is an expansion in v/c therefore it is valid when the separation between the two bodies is not too small (much larger than the size of the bodies) i.e. during the inspiral phase

Newtonian estimate

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\frac{Gm\mu}{r} \qquad \text{i.e.} \qquad \frac{v^2}{c^2} = \frac{R_s}{2r}$$

$$R_s = 2\frac{Gm}{c^2}$$



taken from Blanchet et al. *Phys.Rev.D* **81** (2010)

... that consistently takes into account the non-linearities of the Einstein field equations!

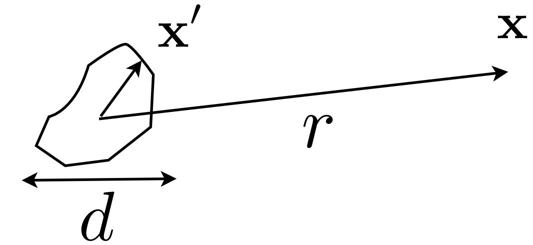
In particular, it is different from linearized theory + multipole expansion which is also an expansion in powers of v/c.

Linearized theory and multipole expansion

Einstein equations

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\xrightarrow[h^2 \to 0]{\text{linearize}}$ $\partial^\mu \bar{h}_{\mu\nu} = 0$

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$



$$h_{ij}^{TT} = \frac{4G}{c^4} \Lambda_{ij,kl} \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} T_{kl}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)$$ outside the source

$$\approx \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl} \int d^3\mathbf{x}' T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\mathbf{n}}{c}, \mathbf{x}'\right)$$ in the wavezone $r \gg d$

$$= \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\mathbf{n}}{c}\right) + i\mathbf{k}\cdot\mathbf{x}'}$$

source $|\mathbf{x}'| < d$

typical freq ω_s $\longrightarrow$ $\omega \frac{\mathbf{x}'\cdot n}{c} \sim \frac{v}{c} \ll 1$

$v \sim \omega_s d$

Expansion $e^{-i\omega\frac{\mathbf{x}'\cdot n}{c}} \approx 1 - \frac{i\omega}{c} x'^i n^i + \frac{1}{2}\left(\frac{i\omega}{c}\right)^2 x'^i x'^j n^i n^j + \quad ...$

Mass quadrupole

Mass octupole
+ current quadrupole $\sim v/c$

But near the source, $h \sim \frac{R_s}{d}$ which, for self gravitating objects, is $\sim \frac{v^2}{c^2}$ $\longrightarrow$ we cannot linearize the Einstein eqs and independently expand in v/c for self gravitating sources

PN approximation scheme (1/3)

rewrite Einstein eqs

$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ $\longrightarrow$

$\partial_\mu h^{\alpha\mu} = 0$ harmonic gauge

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

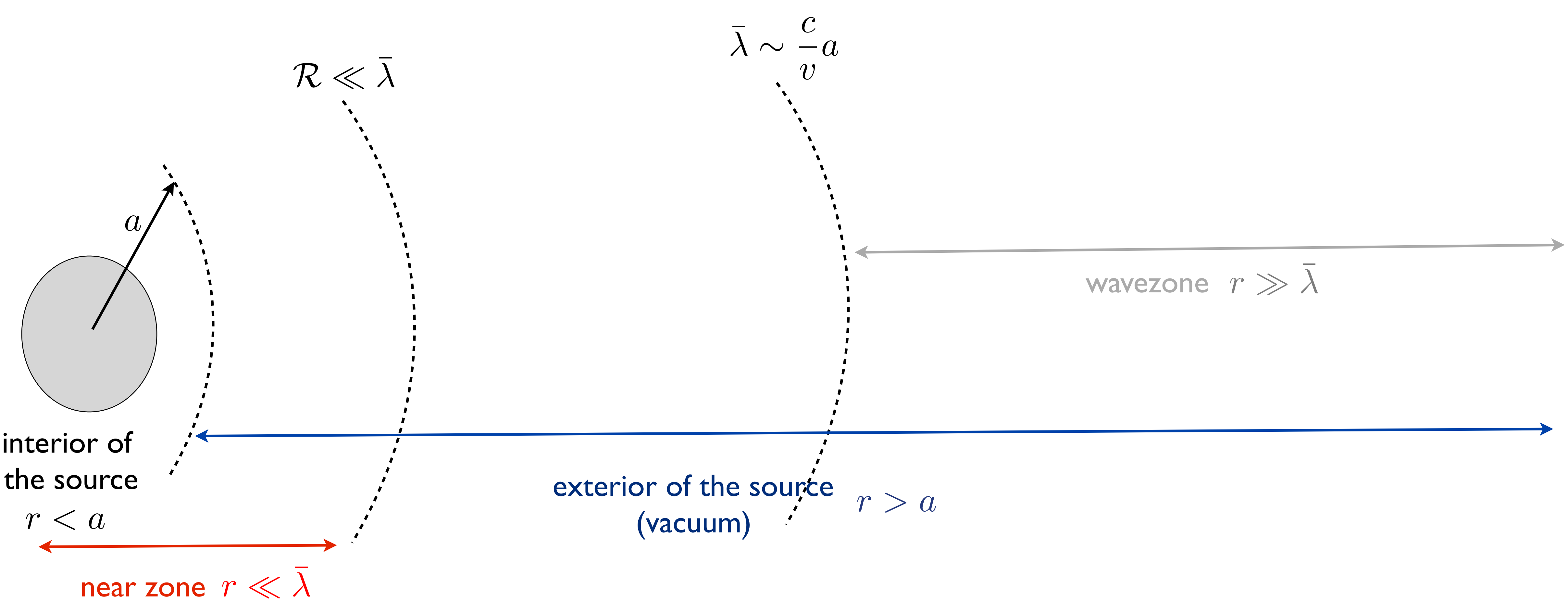
$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ "flat" d'Alembertian

$$\Box^{-1} f = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}'-\mathbf{x}|}{c}\right)$$

$\tau^{\mu\nu}$ stress-energy pseudo tensor of matter + gravitational fields

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}$$

$$\begin{aligned}\Lambda^{\alpha\beta} = & -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ & -g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ & +\frac{1}{8}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}.\end{aligned}$$



PN approximation scheme (2/3)

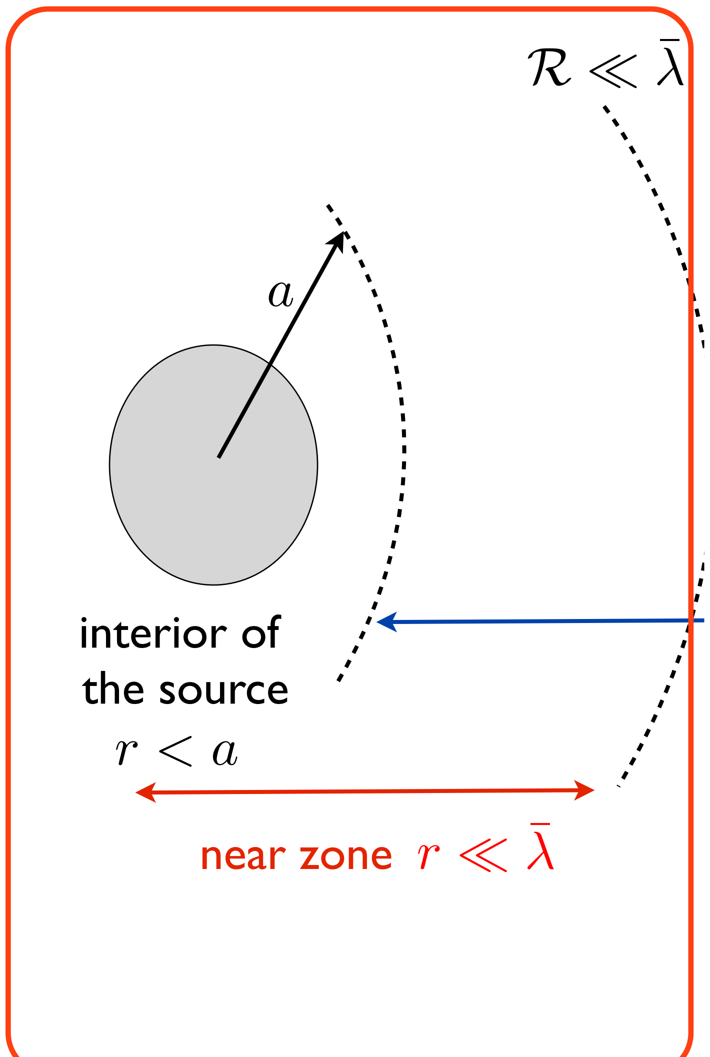
$$\partial_\mu h^{\alpha\mu} = 0$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

Write the solution as formal PN series in powers of 1/c and solve iteratively order by order

$$\Box^{-1} f = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}'-\mathbf{x}|}{c}\right)$$

retardation effects are small we can (PN) expand inside the integrals



Beyond leading order, even if the source has compact support, the support of the integral diverges at spatial infinity... first need for a regularization

In fact, the retarded integral is not exactly the correct solution

How to impose the no incoming radiation condition?
... the definition of the appropriate inverse operator requires knowledge from the far-zone

see e.g. Blanchet's Living Review for a detailed construction of the solution

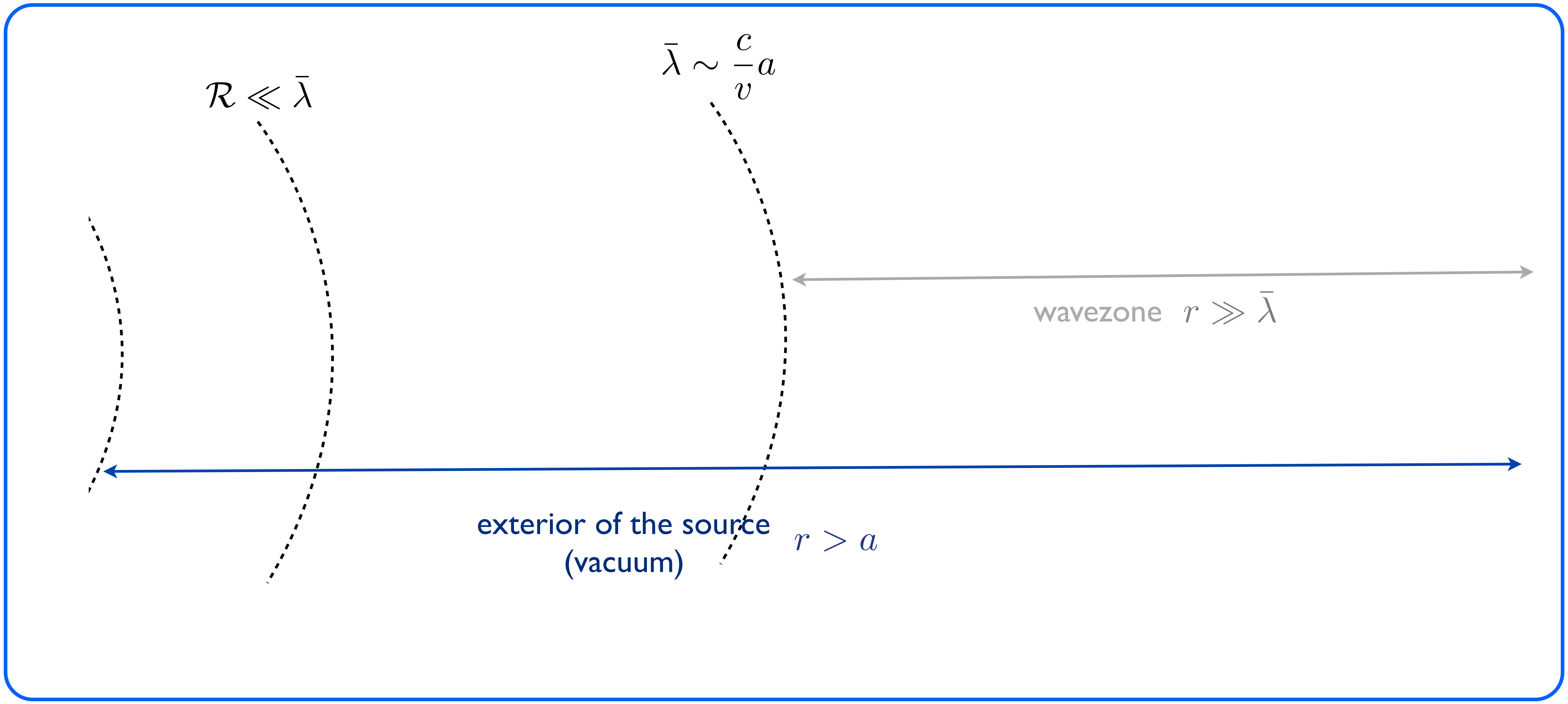
PN approximation scheme (2/3)

$$\partial_\mu h^{\alpha\mu} = 0 \quad \text{harmonic gauge}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

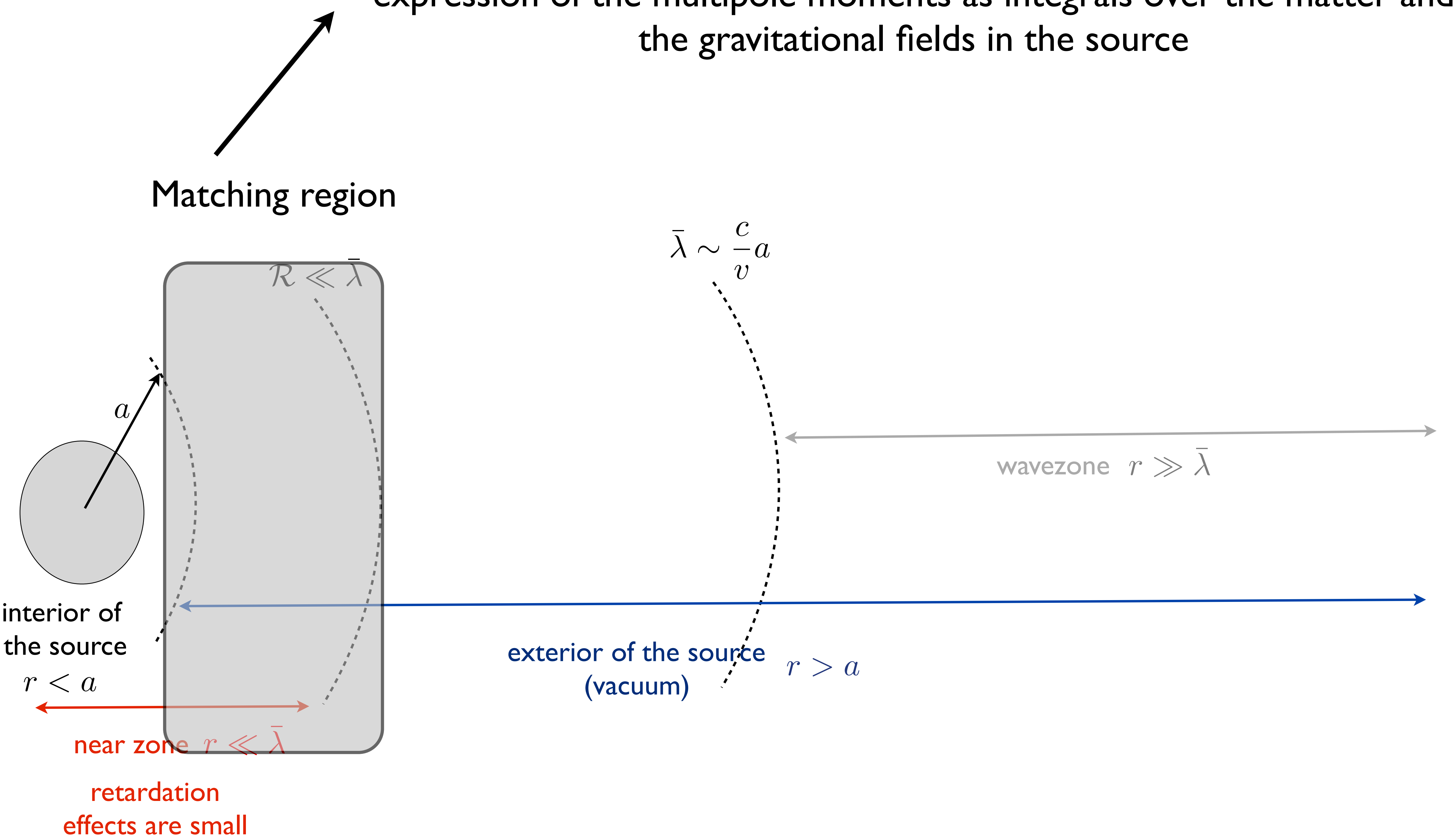
The most general solution in vacuum
can be obtained by combining
post-Minkowskian expansion
+ multipole expansion

$$h^{\mu\nu}_{\text{ext}} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}[I_L, J_L, W_L, X_L, Y_L, Z_L]$$



PN approximation scheme (2/3)

Both expansions are valid. A matching procedure provides an expression of the multipole moments as integrals over the matter and the gravitational fields in the source



PN approximation scheme (3/3)

In practice, the calculation is divided into two (coupled) sub-problems

Computation of the dynamics up to n-th PN order (near-zone resolution of the Einstein eqs)

Newtonian-like equation of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A_{\mathrm{N}}^i + \frac{1}{c^2}A_{1\mathrm{PN}}^i + \frac{1}{c^4}A_{2\mathrm{PN}}^i + \frac{1}{c^5}A_{2.5\mathrm{PN}}^i + \frac{1}{c^6}A_{3\mathrm{PN}}^i + \frac{1}{c^7}A_{3.5\mathrm{PN}}^i + \mathcal{O}(8)$$

quasi-circular orbits in the CM frame $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$

"conserved" Energy

$$E = -\frac{\mu c^2 x}{2}\left[1 + e_1 x + e_2 x^2 + e_3 x^3 + O(1/c^8)\right]$$

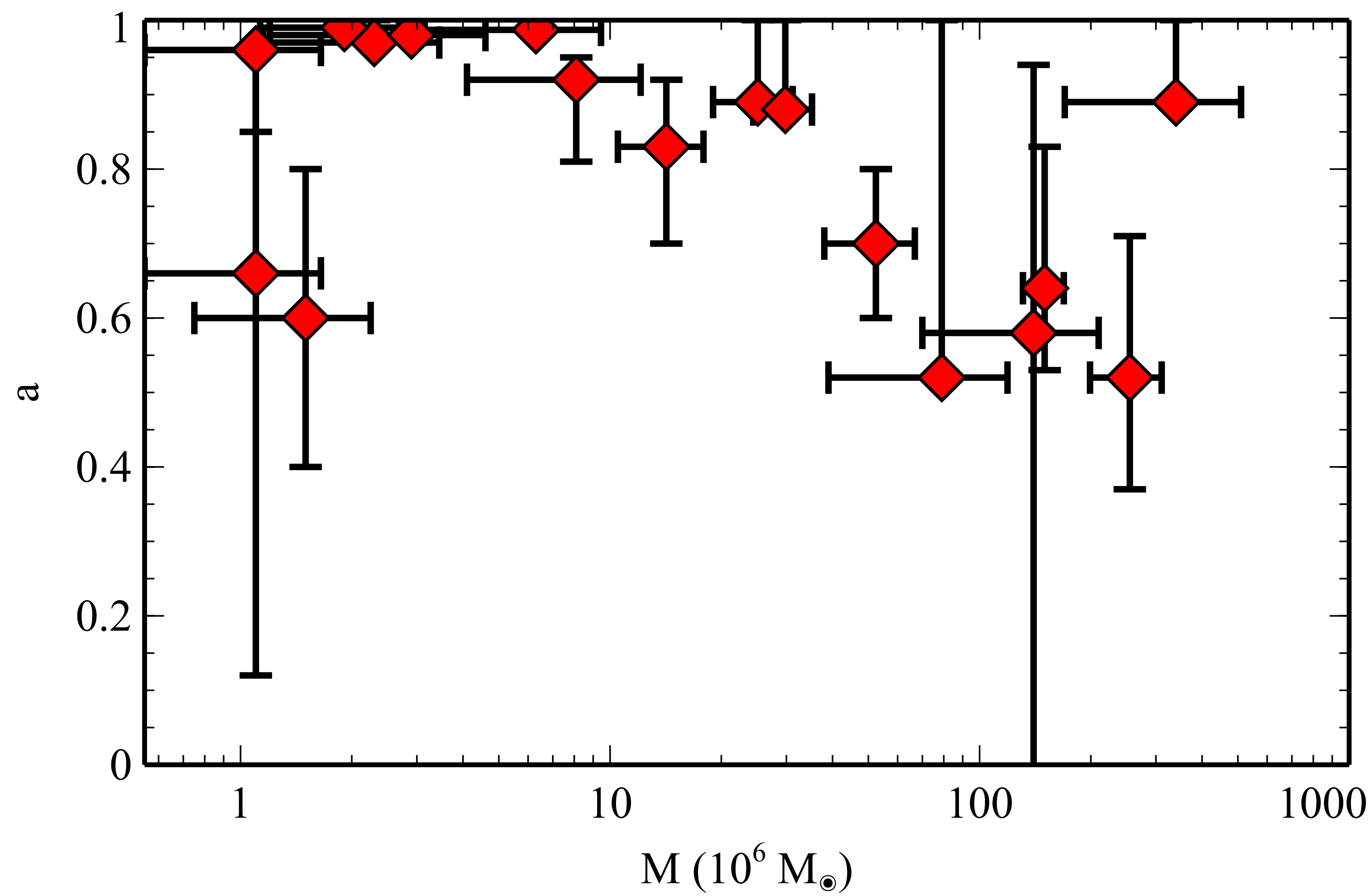
Computation of the radiation up to n-th PN order

flux $$\mathcal{F} = \frac{32c^5}{5G}x^5\nu^2\left[1 + f_1\,x + f_{1.5}\,x^{3/2} + f_2\,x^2 + f_{2.5}\,x^{5/2} + f_3\,x^3 + f_{3.5}\,x^{7/2} + \mathcal{O}(8)\right]$$

Finally, the balance equation $\frac{dE}{dt} = -\mathcal{F}$ provides the phase evolution

Spinning black holes

Recent astrophysical observations indicate that black holes generically have (large!) spins

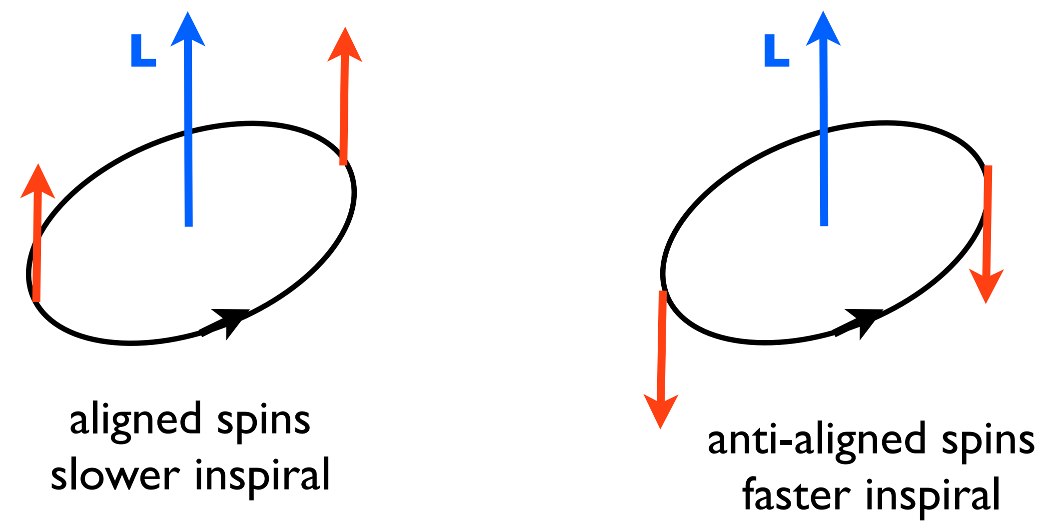


taken from Reynolds astro-ph.HE 1302.3260 (2013)

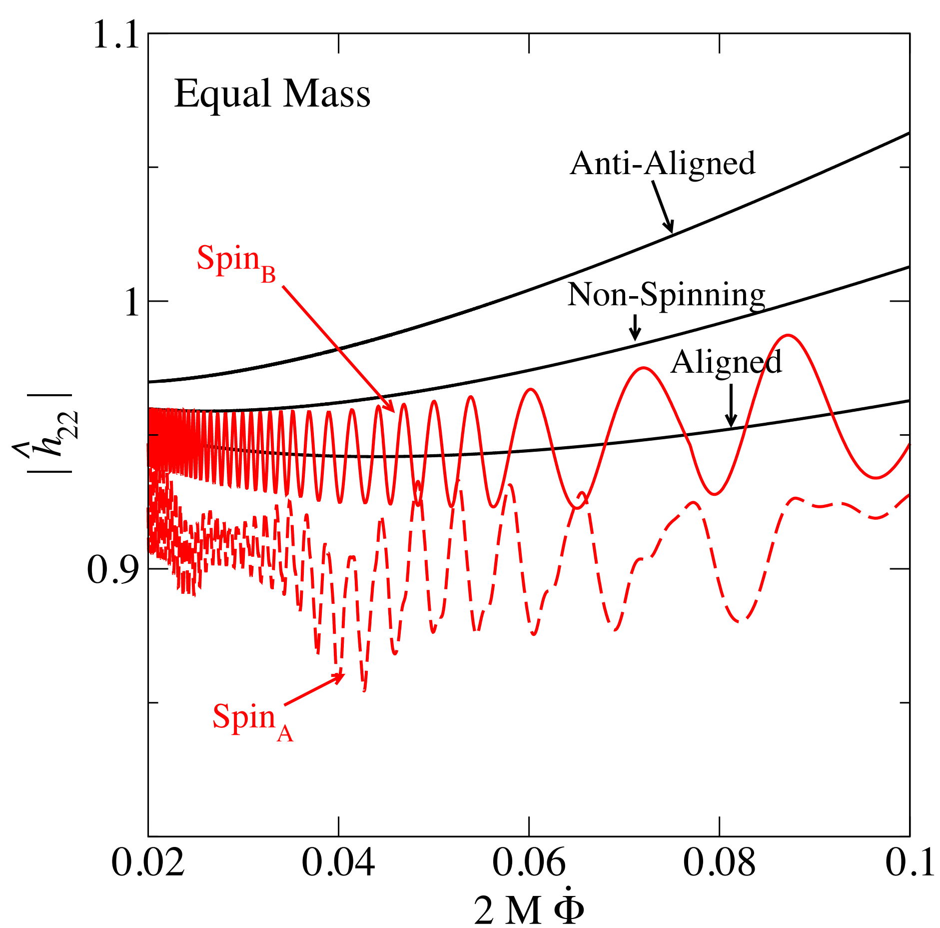
Both Super Massive Black Holes and Stellar Mass Black holes

Effect of the spin on the inspiral

The components of the spins that are orthogonal to the orbital plane change the inspiral rate, i.e. in particular the phase



The components of the spins in the orbital plane cause the orbital plane to precess: strong amplitude modulations



taken from Arun, Buonanno, Faye, Ochsner (09)

Spin ``power counting"

The spin of a rotating compact body is of the order of $S_{\rm true} \sim m\, l\, v_{\rm spin}$ with $l \sim \frac{Gm}{c^2}$

⟶ For maximally rotating bodies, $v_{\rm spin} \sim c$ so $S_{\rm true} \sim \chi \frac{Gm^2}{c}$ is formally 0.5 PN

⟶ For slowly rotating bodies, $v_{\rm spin} \ll c$ so $S_{\rm true}$ is formally 1 PN

We adopt the following spin (re-)definition $S \equiv c\, S_{\rm true} = \chi G m^2$

⟶ For maximally rotating objects, our spin variable is Newtonian

With this definition, the spin enters the Newtonian-like equation of motion at the following orders:

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A^i_{\rm N} + \frac{1}{c^2} A^i_{\rm 1PN} + \frac{1}{c^3} {A_S^i}_{\rm 1.5PN} + \frac{1}{c^4}\left[A^i_{\rm 2PN} + {A_{SS}^i}_{\rm 2PN}\right] + \frac{1}{c^5}\left[A^i_{\rm 2.5PN} + {A_S^i}_{\rm 2.5PN}\right] + \frac{1}{c^6}\left[A^i_{\rm 3PN} + {A_{SS}^i}_{\rm 3PN}\right] + \frac{1}{c^7}\left[A^i_{\rm 3.5PN} + {A_S^i}_{\rm 3.5PN}\right] + \mathcal{O}(8)$$

Progress of the spin PN computations: EOM

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A_{\mathrm{N}}^i + \frac{1}{c^2}A_{1\mathrm{PN}}^i + \frac{1}{c^3}\underset{S}{A^i}_{1.5\mathrm{PN}} + \frac{1}{c^4}\left[A_{2\mathrm{PN}}^i + \underset{SS}{A^i}_{2\mathrm{PN}}\right] + \frac{1}{c^5}\left[A_{2.5\mathrm{PN}}^i + \underset{S}{A^i}_{2.5\mathrm{PN}}\right] + \frac{1}{c^6}\left[A_{3\mathrm{PN}}^i + \underset{SS}{A^i}_{3\mathrm{PN}}\right] + \frac{1}{c^7}\left[A_{3.5\mathrm{PN}}^i + \underset{S}{A^i}_{3.5\mathrm{PN}}\right] + \mathcal{O}(8)$$

Three different formalisms: harmonic gauge, Hamiltonian, Effective Field Theory

LO ($1/c^3$):
Barker and O'Connell (75, 79)
Goldberger, Rothstein (06) (EFT approach)

NLO ($1/c^5$):
Tagoshi, Ohashi, Owen (98, 01)
Blanchet, Buonanno, Faye (06)
Damour, Jaranowski, Schäfer, (08) (ADM formalism)
Levi (10), Porto (10) (EFT approach)

Spin-Spin effects:

LO ($1/c^4$): Goldberger, Rothstein (06) (EFT approach)
NLO ($1/c^6$): Steinhoff, Hergt, Schäfer (08,10) (ADM)
Porto, Rothstein (10), Levi (11) (EFT)
NNLO ($1/c^8$) spin1-spin2:
Levi (12) (EFT)
Hartung, Steinhoff (11) (ADM)

Here we compute the 3.5PN spin-orbit (linear in spin) correction together with the evolution equations for the spins

NNLO ($1/c^7$):
Hartung Steinhoff (11) (ADM coords)
Marsat, Bohe, Faye, Blanchet, (12)

Progress of the spin PN computations: Radiation

So far, a wave generation formalism has only been derived in the harmonic gauge formulation (although EFT on the way (cf Porto (06))

$$\mathcal{F} = \frac{32c^5}{5G} x^5 \nu^2 \left[1 + f_1\, x + f_{1.5}\, x^{3/2} + f_2\, x^2 + f_{2.5}\, x^{5/2} + f_3\, x^3 + f_{3.5}\, x^{7/2} + \mathcal{O}(4) \right]$$

For the flux

Spin-Orbit effects

LO ($1/c^3$): Kidder, Will, Wiseman (93, 95)

NLO ($1/c^5$): Blanchet, Buonanno, Faye (06)

NNLO ($1/c^7$): Bohe, Marsat, Blanchet, (13)

Tail SO effects

LO ($1/c^6$): Blanchet, Buonanno, Faye (06)

NLO ($1/c^8$): Marsat, Bohe, Blanchet, (in preparation)

Spin-Spin effects

LO ($1/c^4$): Mikoczi, Vasuth, Gergely (05)

For the polarizations

SO LO ($1/c^3$): Kidder, Will, Wiseman (93, 95)
Arun, Buonanno, Faye, Ochsner (09)

SS LO ($1/c^4$): Kidder, Will, Wiseman (95, 96) Spin1-Spin2
Buonanno, Faye, Hinderer Spin1-Spin1

tail LO ($1/c^6$): Blanchet, Buonanno, Faye (06)

Progress

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Description of the system: effective pole-dipole formalism

Description in terms of point particles

$$T_{\mathrm{M}}^{\mu\nu} = c^2 \int_{-\infty}^{+\infty} \mathrm{d}\tau \; p^{(\mu} u^{\nu)} \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g(x)}} \, ,$$

Model developped by
Mathisson, Papapetrou, Tulczyjew
generalized by Dixon, Bailey & Israel

$$u^\mu = \mathrm{d}y^\mu / (c\mathrm{d}\tau)$$
$$u_\mu u^\mu = -1$$

Effective approach: write the most general stress-energy tensor built with (derivatives of) delta functions

$$T_{\mathrm{D}}^{\mu\nu} = -c \int_{-\infty}^{+\infty} \mathrm{d}\tau \; \nabla_\rho \left[S^{\rho(\mu} u^{\nu)} \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g(x)}} \right] \, .$$

$S^{\mu\nu}(t)$ is an antisymmetric tensor encoding the spin

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = c^2 \left(p^\mu u^\nu - p^\nu u^\mu \right) \, ,$$
$$\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma}$$

Mathisson-Papapetrou equations

$$\frac{\mathrm{D}}{\mathrm{d}\tau} \equiv c u^\nu \nabla_\nu$$

Effective pole-dipole formalism: simplifications

Supplementary spin condition $S^{\mu\nu} p_\nu = 0$
+
Only spin orbit effects

$\longrightarrow$ $p^\mu = m\, c\, u^\mu + \mathcal{O}(S^2)$

The stress energy-tensor reduces to

$$T^{\mu\nu}_{\rm M} = m u^0 v^\mu v^\nu \frac{\delta^{(3)}(\mathbf{x}-\mathbf{y}(t))}{\sqrt{-g(t,\mathbf{x})}} \,,$$

$$T^{\mu\nu}_{\rm D} = -\frac{1}{c}\nabla_\rho \left[S^{\rho(\mu} v^{\nu)} \frac{\delta^{(3)}(\mathbf{x}-\mathbf{y}(t))}{\sqrt{-g(t,\mathbf{x})}} \right]$$

$$y^\mu = (c\, t, \mathbf{y}(t))$$

$$v^\mu = c u^\mu / u^0$$

$$u^0 = 1/\sqrt{-g_{\rho\sigma} v^\rho v^\sigma / c^2}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = \mathcal{O}(S^2) \,,$$

$$mc \frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma} + \mathcal{O}(S^2)$$

Equations of motion

Note that using the SSC, we can work with the spatial components S^{ij} of $S^{\mu\nu}$

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PN iteration of the Einstein's equations in harm gauge

In the harmonic (De Donder) gauge

$$\partial_\nu(\sqrt{-g}g^{\mu\nu}) = 0$$

the metric can be conveniently parametrized via a set of potentials which satisfy flat-space wave equations sourced by $T^{\mu\nu}$ and the lower order potentials.

$$\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_i V_i + \frac{V^3}{6}\right) \\
&+ \frac{32}{c^8}\left(\hat{T} - \frac{1}{2}V\hat{X} + \hat{R}_i V_i - \frac{1}{2}V V_i V_i - \frac{V^4}{48}\right) + \mathcal{O}(10)\,, \\
g_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}V^2 V_i\right) + \mathcal{O}(9)\,, \\
g_{ij} &= \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_k V_k + \frac{V^3}{6}\right)\right] \\
&+ \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \frac{1}{2}V\hat{W}_{ij} - V_i V_j\right) + \mathcal{O}(8)
\end{aligned}$$

For instance

$$\begin{aligned}
V &= \Box_{\mathcal{R}}^{-1}[-4\pi G\,\sigma]\,, \\
\hat{W}_{ij} &= \Box_{\mathcal{R}}^{-1}\left[-4\pi G\left(\sigma_{ij} - \delta_{ij}\sigma_{kk}\right) - \partial_i V \partial_j V\right]
\end{aligned}$$

$$\begin{aligned}
\sigma &= \frac{1}{c^2}(T^{00} + T^{ii}) \\
\sigma_i &= \frac{1}{c}T^{0i} \\
\sigma_{ij} &= T^{ij}
\end{aligned}$$

with

$$(\Box_{\mathcal{R}}^{-1}f)(\mathbf{x},t) = -\frac{1}{4\pi}\int\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

Computation of the potentials

Expand the retardations inside the integrals

$$-4\pi\,\Box^{-1}_{\mathcal{R}} f(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$
$$= \int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}',t) - \frac{1}{c}\int \mathrm{d}^3\mathbf{x}' \partial_t f(\mathbf{x}',t) + \frac{1}{2c^2}\int \mathrm{d}^3\mathbf{x}' |\mathbf{x}-\mathbf{x}'| \partial_t^2 f(\mathbf{x}',t) + \mathcal{O}(3)$$

⟶ only valid in the near-zone $r \ll \lambda$

Carefully treat the distributional derivatives appearing in the sources

$$\partial_{ij} f = \partial^{\mathrm{ord}}_{ij} f + D_i\left[\partial^{\mathrm{ord}}_j f\right] + \partial^{\mathrm{ord}}_i D_j[f]$$

$$D_i\left(\frac{n^L}{r^m}\right) = 4\pi \frac{(-)^m 2^m (\ell+1)!(\frac{\ell+m-1}{2})!}{(\ell+m)!} \sum_{p=p_0}^{[m/2]} \frac{\Delta^{p-1}\partial_{(M-2P}\,\delta_{iL+2P-M)}}{2^{2p}(p-1)!(m-2p)!(\frac{\ell+1-m}{2}+p)!}$$

when l+m is even
0 otherwise

e.g. $\Delta \frac{1}{r} = -4\pi\delta$

Gel'Fand-Shilov formula for homogeneous functions

Complexity of the source terms

Compute the spin part of the following potentials 1PN order beyond previous calculations:

$$V = \Box_{\mathcal{R}}^{-1}[-4\pi G\,\sigma]\,,$$
$$V_i = \Box_{\mathcal{R}}^{-1}[-4\pi G\,\sigma_i]\,,$$
$$\hat{X} = \Box_{\mathcal{R}}^{-1}\left[-4\pi G\,V\sigma_{ii} + \hat{W}_{ij}\partial_{ij}V + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i\right]\,,$$
$$\hat{R}_i = \Box_{\mathcal{R}}^{-1}\left[-4\pi G\,(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right]\,,$$
$$\hat{W}_{ij} = \Box_{\mathcal{R}}^{-1}\left[-4\pi G\,(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V\right]\,,$$

compact support term computed in all space

"quadratic" term computed in all space

And the leading order spin contribution to these "new" potentials:

$$\hat{T} = \Box_{\mathcal{R}}^{-1}\left[-4\pi G\left(\frac{1}{4}\sigma_{ij}\hat{W}_{ij} + \frac{1}{2}V^2\sigma_{ii} + \sigma V_i V_i\right) + \hat{Z}_{ij}\partial_{ij}V + \hat{R}_i\partial_t\partial_i V - 2\partial_i V_j\partial_j\hat{R}_i - \partial_i V_j\partial_t\hat{W}_{ij} + VV_i\partial_t\partial_i V + 2V_i\partial_j V_i\partial_j V + \frac{3}{2}V_i\partial_t V\partial_i V + \frac{1}{2}V^2\partial_t^2 V + \frac{3}{2}V(\partial_t V)^2 - \frac{1}{2}(\partial_t V_i)^2\right]\,,$$
$$\hat{Y}_i = \Box_{\mathcal{R}}^{-1}\left[-4\pi G\left(-\sigma\hat{R}_i - \sigma VV_i + \frac{1}{2}\sigma_k\hat{W}_{ik} + \frac{1}{2}\sigma_{ik}V_k + \frac{1}{2}\sigma_{kk}V_i\right) + \hat{W}_{kl}\partial_{kl}V_i - \partial_t\hat{W}_{ik}\partial_k V + \partial_i\hat{W}_{kl}\partial_k V_l - \partial_k\hat{W}_{il}\partial_l V_k - 2\partial_k V\partial_i\hat{R}_k - \frac{3}{2}V_k\partial_i V\partial_k V - \frac{3}{2}V\partial_t V\partial_i V - 2V\partial_k V\partial_k V_i + V\partial_t^2 V_i + 2V_k\partial_k\partial_t V_i\right]\,,$$
$$\hat{Z}_{ij} = \Box_{\mathcal{R}}^{-1}\left[-4\pi G\,V(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - 2\partial_{(i}V\partial_t V_{j)} + \partial_i V_k\partial_j V_k + \partial_k V_i\partial_k V_j - 2\partial_{(i}V_k\partial_k V_{j)} - \delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k) - \frac{3}{4}\delta_{ij}(\partial_t V)^2\right]\,.$$

"cubic" term cannot be computed in all space but just at the location of the particles

$$(\Box_{\mathcal{R}}^{-1}f)(\mathbf{x},t) = -\frac{1}{4\pi}\int\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}f\left(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

The EOM in terms of the potentials

The EOM $mc\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}u^\nu S^{\rho\sigma}$ can be rewritten $\frac{\mathrm{d}P_i}{\mathrm{d}t} = F_i + \mathcal{F}_i$ with

$$\begin{aligned}
P_i &= v^i \\
&+ \frac{1}{c^2}\left(\frac{1}{2}v^2 v^i + 3Vv^i - 4V_i\right) \\
&+ \frac{1}{c^4}\left(\frac{3}{8}v^4 v^i + \frac{7}{2}Vv^2 v^i - 4V_j v^i v^j - 2V_i v^2\right. \\
&\qquad \left. + \frac{9}{2}V^2 v^i - 4VV_i + 4\hat{W}_{ij}v^j - 8\hat{R}_i\right) \\
&+ \frac{1}{c^6}\left(\frac{5}{16}v^6 v^i + \frac{33}{8}Vv^4 v^i - \frac{3}{2}V_i v^4 - 6V_j v^i v^j v^2 + \frac{49}{4}V^2 v^2 v^i\right. \\
&\qquad + 2\hat{W}_{ij}v^j v^2 + 2\hat{W}_{jk}v^i v^j v^k - 10VV_i v^2 - 20VV_j v^i v^j \\
&\qquad - 4\hat{R}_i v^2 - 8\hat{R}_j v^i v^j + \frac{9}{2}V^3 v^i + 12V_j V_j v^i + 12\hat{W}_{ij}Vv^j \\
&\qquad + 12\hat{X}v^i + 16\hat{Z}_{ij}v^j - 10V^2 V_i \\
&\qquad \left. -8\hat{W}_{ij}V_j - 8V\hat{R}_i - 16\hat{Y}_i\right) + \mathcal{O}(8)\,,
\end{aligned}$$

$$\begin{aligned}
F_i &= \partial_i V \\
&+ \frac{1}{c^2}\left(-V\,\partial_i V + \frac{3}{2}\partial_i V\,v^2 - 4\partial_i V_j\,v^j\right) \\
&+ \frac{1}{c^4}\left(\frac{7}{8}\partial_i V\,v^4 - 2\partial_i V_j\,v^j v^2 + \frac{9}{2}V\,\partial_i V\,v^2 + 2\partial_i\hat{W}_{jk}\,v^j v^k - 4V_j\,\partial_i V\,v^j\right. \\
&\qquad \left. - 4V\,\partial_i V_j\,v^j - 8\partial_i\hat{R}_j\,v^j + \frac{1}{2}V^2\,\partial_i V + 8V_j\,\partial_i V_j + 4\partial_i\hat{X}\right) \\
&+ \frac{1}{c^6}\left(\frac{11}{16}v^6\partial_i V - \frac{3}{2}\partial_i V_j\,v^j v^4 + \frac{49}{8}V\,\partial_i V\,v^4 + \partial_i\hat{W}_{jk}\,v^2 v^j v^k\right. \\
&\qquad -10V_j\,\partial_i V\,v^2 v^j - 10V\partial_i V_j\,v^2 v^j - 4\partial_i\hat{R}_j\,v^2 v^j + \frac{27}{4}V^2\,\partial_i V\,v^2 \\
&\qquad +12V_j\partial_i V_j\,v^2 + 6\hat{W}_{jk}\,\partial_i V\,v^j v^k + 6V\,\partial_i\hat{W}_{jk}\,v^j v^k + 6\partial_i\hat{X}v^2 \\
&\qquad +8\partial_i\hat{Z}_{jk}\,v^j v^k - 20V_j V\partial_i V v^j - 10V^2\,\partial_i V_j\,v^j - 8V_k\,\partial_i\hat{W}_{jk}\,v^j \\
&\qquad -8\hat{W}_{jk}\,\partial_i V_k\,v^j - 8\hat{R}_j\,\partial_i V\,v^j - 8V\,\partial_i\hat{R}_j\,v^j - 16\partial_i\hat{Y}_j\,v^j \\
&\qquad -\frac{1}{6}V^3\,\partial_i V - 4V_j\,V_j\,\partial_i V + 16\hat{R}_j\,\partial_i V_j + 16V_j\,\partial_i\hat{R}_j \\
&\qquad \left. -8V\,V_j\,\partial_i V_j - 4\hat{X}\,\partial_i V - 4V\,\partial_i\hat{X} + 16\partial_i\hat{T}\right) + \mathcal{O}(8)\,.
\end{aligned}$$

$$m\mathcal{F}^i = \frac{1}{c^3}f_3^i + \frac{1}{c^5}f_5^i + \frac{1}{c^7}f_7^i + \mathcal{O}(9)$$

$$\begin{aligned}
f_3^i &= S^{ij}\left(\partial_t\partial_j V + v^k\partial_{jk}V\right) + S^{jk}\left(2v^k\partial_{ij}V - 2\partial_{ij}V_k\right)\,, \\
f_5^i &= S^{ij}\left(-v^k\partial_j V\partial_k V + 2Vv^k\partial_k\partial_j V + 2\partial_j V_k\partial_k V + 2v^j\partial_k V\partial_k V - 2\partial_k V_j\partial_k V\right. \\
&\qquad \left. +\partial_j V\partial_t V + 2V\partial_t\partial_j V + v^j v^k\partial_t\partial_k V + v^j\partial_t^2 V\right) \\
&\quad + S^{jk}\left(4v^j\partial_i V\partial_k V - 4\partial_j V\partial_k V_i - 4\partial_i V\partial_k V_j + 4\partial_{ik}\hat{R}_j - 4Vv^j\partial_{ik}V\right. \\
&\qquad + 4V_j\partial_{ik}V + 2v^j v^l\partial_{ik}V_l - 2v^l\partial_{ik}\hat{W}_{lj} - 2v^j v^l\partial_{kl}V_i + 2v^l\partial_{kl}\hat{W}_{ij} \\
&\qquad \left. -2v^j\partial_t\partial_i V_k - 2v^j\partial_t\partial_k V_i + 2\partial_t\partial_k\hat{W}_{ij}\right)\,, \\
f_7^i &= S^{ij}\left(4\partial_j\hat{R}_k\partial_k V - 2Vv^k\partial_j V\partial_k V + 4V_k\partial_j V\partial_k V + 8v^k\partial_j V_l\partial_k V_l + 2V^2v^k\partial_{kj}V\right. \\
&\qquad + 8v^k V_l\partial_{kj}V_l + 4v^k\partial_{kj}\hat{X} - 4\partial_k V\partial_k\hat{R}_j + 4Vv^j\partial_k V\partial_k V - 4V_j\partial_k V\partial_k V \\
&\qquad - 2v^k\partial_j\hat{W}_{kl}\partial_l V + 2v^j v^k\partial_k V_l\partial_l V - 2v^k\partial_k\hat{W}_{jl}\partial_l V - 2v^j v^k\partial_l V_k\partial_l V + 2v^k\partial_l\hat{W}_{kj}\partial_l V \\
&\qquad + 2V\partial_j V\partial_t V - 2v^k\partial_j V_k\partial_t V + v^j v^k\partial_k V\partial_t V - 2v^k\partial_k V_j\partial_t V + v^j\left(\partial_t V\right)^2 \\
&\qquad + 8\partial_j V_k\partial_t V_k + 4v^j\partial_k V\partial_t V_k - 2\partial_k V\partial_t\hat{W}_{jk} + 2V^2\partial_t\partial_j V + 8V_k\partial_t\partial_j V_k \\
&\qquad \left. +4\partial_t\partial_j\hat{X} + 6Vv^j v^k\partial_t\partial_k V - 4v^k V_j\partial_t\partial_k V + 6Vv^j\partial_t^2 V - 4V_j\partial_t^2 V\right) \\
&\quad + S^{jk}\left(-8V\partial_i V_k\partial_j V - 8\partial_i V\partial_k\hat{R}_j + 8Vv^j\partial_i V\partial_k V - 8V_j\partial_i V\partial_k V - 4v^j v^l\partial_i V_l\partial_k V\right. \\
&\qquad + 8\partial_j\hat{R}_i\partial_k V + 8V\partial_i V\partial_k V_j + 8v^l\partial_i V_l\partial_k V_j + 4v^l\partial_i V_j\partial_k V_l - 12v^j\partial_i V_l\partial_k V_l \\
&\qquad - 4v^l\partial_j V_i\partial_k V_l + 4\partial_j V_l\partial_k\hat{W}_{il} + 4\partial_i V_l\partial_k\hat{W}_{jl} + 4v^j v^l\partial_{ki}\hat{R}_l + 8\hat{R}_j\partial_{ki}V \\
&\qquad - 4V^2v^j\partial_{ki}V + 8VV_j\partial_{ki}V + 4v^j v^l V_l\partial_{ki}V - 8v^l\hat{W}_{lj}\partial_{ki}V + 4V^2\partial_{ki}V_j \\
&\qquad + 8v^l V_l\partial_{ki}V_j + 8Vv^j v^l\partial_{ki}V_l - 16v^j V_l\partial_{ki}V_l + 4\hat{W}_{jl}\partial_{ki}V_l - 4Vv^l\partial_{ki}\hat{W}_{lj} \\
&\qquad + 4V_l\partial_{ki}\hat{W}_{lj} - 8v^j\partial_{ki}\hat{X} + 8\partial_{ki}\hat{Y}_j - 8v^l\partial_{ki}\hat{Z}_{lj} - 4v^j v^l\partial_i V_k\partial_l V \\
&\qquad + 4v^j v^l\partial_k V\partial_l V_i - 8v^l\partial_k V_j\partial_l V_i + 4v^j v^l\partial_i V\partial_l V_k - 4v^l\partial_i V_j\partial_l V_k - 4v^j\partial_i V_l\partial_l V_k \\
&\qquad + 4v^l\partial_j V_i\partial_l V_k - 4\partial_j V_l\partial_l\hat{W}_{ik} - 4v^j v^l\partial_{lk}\hat{R}_i + 4v^l\hat{W}_{ij}\partial_{lk}V - 8Vv^j v^l\partial_{lk}V_i \\
&\qquad - 8v^l V_i\partial_{lk}V_j + 4Vv^l\partial_{lk}\hat{W}_{ij} + 8v^l\partial_{lk}\hat{Z}_{ij} + 4v^j\partial_i\hat{W}_{kl}\partial_l V + 4v^j\partial_k\hat{W}_{il}\partial_l V \\
&\qquad - 4v^j\partial_l\hat{W}_{ik}\partial_l V - 4v^j\partial_k V_l\partial_l V_i + 4v^j\partial_l V_k\partial_l V_i - 4\partial_i\hat{W}_{kl}\partial_l V_j - 4\partial_k\hat{W}_{il}\partial_l V_j \\
&\qquad + 4\partial_l\hat{W}_{ik}\partial_l V_j + 4v^j\partial_k V\partial_t V_i - 8\partial_k V_j\partial_t V_i + 4v^j\partial_i V\partial_t V_k + 8\partial_j V_i\partial_t V_k \\
&\qquad - 4\partial_j V\partial_t\hat{W}_{ik} - 4v^j\partial_t\partial_i\hat{R}_k + 4v^j V_k\partial_t\partial_i V - 8Vv^j\partial_t\partial_i V_k + 8V_j\partial_t\partial_i V_k \\
&\qquad + 2v^j v^l\partial_t\partial_i\hat{W}_{lk} - 4v^j\partial_t\partial_k\hat{R}_i + 4\hat{W}_{ij}\partial_t\partial_k V - 8Vv^j\partial_t\partial_k V_i - 8V_i\partial_t\partial_k V_j \\
&\qquad \left. +4V\partial_t\partial_k\hat{W}_{ij} + 8\partial_t\partial_k\hat{Z}_{ij} - 2v^j v^l\partial_t\partial_l\hat{W}_{ik} - 2v^j\partial_t^2\hat{W}_{ik}\right)\,.
\end{aligned}$$

and to compute the acceleration of body 1, one needs to evaluate all these potentials (and their derivatives) at the position of the body 1...

but because of our description of the bodies as point particles, everything diverges at these two points

⟶ Such a description has to be supplemented with a regularization procedure

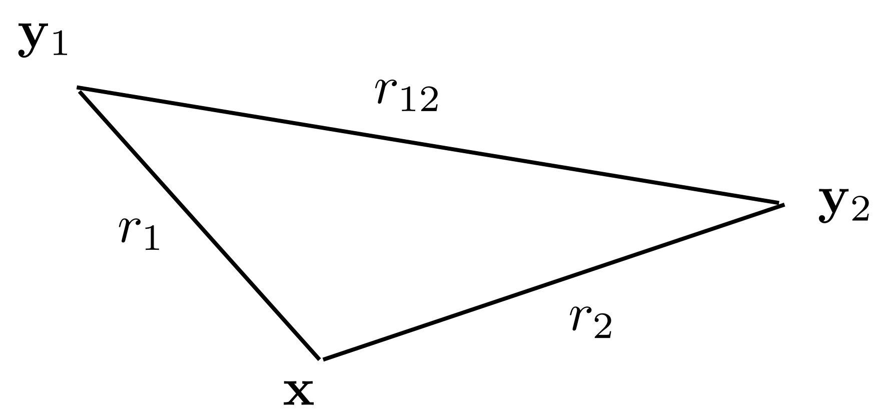
Hadamard regularization of the potentials

Simplest example

$$V = \Box^{-1}_{\mathcal{R}}[-4\pi G\,\sigma] \qquad \text{with} \qquad \sigma = \frac{1}{c^2}(T^{00}+T^{ii}) = m_1\delta_1 + m_2\delta_2 + \mathcal{O}(2)$$

$$V(\mathbf{x},t) = -\frac{1}{4\pi}\int\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}',t) + \mathcal{O}(2)$$

$$= \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \mathcal{O}(2)$$



diverges when $r_1 \to 0$

V generalizes the Newtonian potential so the "correct value" at $r_1=0$ is $(V)_1 = \frac{Gm_2}{r_{12}} + \mathcal{O}(2)$

Hadamard regularization

For a function F singular at $\mathbf{y}_1$ and $\mathbf{y}_2$ which admits an expansion of the form

$$F(\mathbf{x}) = \sum_{a_0\leq a\leq n} r_1^a f_a(\mathbf{n}_1) + o(r_1^n) \qquad \text{we define} \qquad (F)_1 = \int\frac{d\Omega_1}{4\pi}f_0(\mathbf{n}_1)$$

Beware that, in general, $(FG)_1 \neq (F)_1(G)_1$

Hadamard regularization of the potentials

We also need to give a meaning to integrals such as $\int F(\mathbf{x}')\delta_1(\mathbf{x}')\mathrm{d}^3\mathbf{x}'$ and $\int F(\mathbf{x}')\mathrm{d}^3\mathbf{x}'$ for a function F that diverges at the location of the two bodies.

- We use $\int \mathrm{d}^3\mathbf{x}' F(\mathbf{x}')\delta_1 = (F)_1$

  not "standard" distribution theory ⟶ careful in treating derivatives...

- For non-compact support integrals

$$\mathrm{Pf}_{s_1,s_2}\int d^3\mathbf{x}\ F = \lim_{s\to 0}\left\{\int_{\mathbb{R}^3\setminus\mathcal{B}_1(s)\cup\mathcal{B}_2(s)} d^3\mathbf{x}\ F + \sum_{a+3<0}\frac{s^{a+3}}{a+3}\int d\Omega_1 f_a + \ln\left(\frac{s}{s_1}\right)\int d\Omega_1 f_{-3} + 1 \leftrightarrow 2\right\}$$

↖ ambiguities appear but here they vanish in the final result

If the ambiguities remain, one needs to resort to dimensional regularization (much heavier! cf 3PN non-spinning dynamics). Here, we have computed the ambiguous term using dimensional regularization to check our result.

Result

We obtain
$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A_{\mathrm{N}}^i + \frac{1}{c^2}A_{1\mathrm{PN}}^i + \frac{1}{c^3}\underset{S}{A}{}^i{}_{1.5\mathrm{PN}} + \frac{1}{c^4}\left[A_{2\mathrm{PN}}^i + \underset{SS}{A}{}^i{}_{2\mathrm{PN}}\right] + \frac{1}{c^5}\left[A_{2.5\mathrm{PN}}^i + \underset{S}{A}{}^i{}_{2.5\mathrm{PN}}\right] + \frac{1}{c^6}\left[A_{3\mathrm{PN}}^i + \underset{SS}{A}{}^i{}_{3\mathrm{PN}}\right] + \frac{1}{c^7}\left[A_{3.5\mathrm{PN}}^i + \underset{S}{A}{}^i{}_{3.5\mathrm{PN}}\right] + \mathcal{O}(8)$$

with
$$m_1 \underset{S}{A}{}^i_{3.5\mathrm{PN}} = \frac{G}{r_{12}^3}\left[\gamma_{0,1}^i m_2 + \gamma_{1,0}^i m_1\right] + \frac{G^2}{r_{12}^4}\left[\gamma_{0,2}^i m_2^2 + \gamma_{1,1}^i m_1 m_2 + \gamma_{2,0}^i m_1^2\right] + \frac{G^3}{r_{12}^5}\left[\gamma_{0,3}^i m_2^3 + \gamma_{1,2}^i m_1 m_2^2 + \gamma_{2,1}^i m_1^2 m_2 + \gamma_{3,0}^i m_1^3\right]$$

and for example
$$\begin{aligned}
\gamma_{1,0}^i = S_2^{ij} n_{12}^j \Big[& \frac{105}{4}(n_{12}v_{12})(n_{12}v_2)^4 + 15(n_{12}v_{12})(n_{12}v_2)^2(v_{12}v_2) - 15(n_{12}v_2)^3(v_{12}v_2) \\
& -6(n_{12}v_2)(v_{12}v_2)^2 - \frac{15}{2}(n_{12}v_{12})(n_{12}v_2)^2 v_2^2 - 3(n_{12}v_{12})(v_{12}v_2)v_2^2 \\
& +3(n_{12}v_2)(v_{12}v_2)v_2^2 - \frac{3}{4}(n_{12}v_{12})v_2^4\Big] \\
& + S_2^{ij} v_{12}^j \left[-\frac{15}{2}(n_{12}v_2)^4 - 6(n_{12}v_2)^2(v_{12}v_2) + 3(n_{12}v_2)^2 v_2^2 + 2(v_{12}v_2)v_2^2 + \frac{1}{2}v_2^4\right] \\
& + (S_2 n_{12} v_{12}) n_{12}^i \left[\frac{105}{4}(n_{12}v_2)^4 + 15(n_{12}v_2)^2(v_{12}v_2) - \frac{15}{2}(n_{12}v_2)^2 v_2^2 - 3(v_{12}v_2)v_2^2 - \frac{3}{4}v_2^4\right] \\
& + (S_2 n_{12} v_{12}) v_{12}^i \left[15(n_{12}v_{12})(n_{12}v_2)^2 + 15(n_{12}v_2)^3 - 3(n_{12}v_{12})v_2^2 - 3(n_{12}v_2)v_2^2\right] \\
& + (S_2 n_{12} v_2) v_{12}^i \left[-15(n_{12}v_{12})(n_{12}v_2)^2 - 6(n_{12}v_{12})(v_{12}v_2) + 6(n_{12}v_2)(v_{12}v_2) - 3(n_{12}v_{12})v_2^2\right] \\
& + (S_2 n_{12} v_2) v_2^i \left[-15(n_{12}v_{12})(n_{12}v_2)^2 - 6(n_{12}v_{12})(v_{12}v_2) + 6(n_{12}v_2)(v_{12}v_2) - 3(n_{12}v_{12})v_2^2\right] \\
& + (S_2 v_{12} v_2) v_{12}^i \left[6(n_{12}v_2)^2 + 4(v_{12}v_2) + 2v_2^2\right] \\
& + (S_2 v_{12} v_2) v_2^i \left[6(n_{12}v_2)^2 + 4(v_{12}v_2) + 2v_2^2\right]
\end{aligned}$$

Tests of the result

- Existence of 10 conserved integrals of the motion
  (when neglecting radiation reaction terms)

  Energy, Linear Momentum, Angular Momentum, Center of Mass Position

- Lorentz invariance

  The harmonic gauge condition is manifestly Lorentz invariant so our equation of motion must take the same form in two frames related to one another by a boost

- Test-mass limit

  Recover the motion of a test mass around Kerr and of a spinning test mass around Schwarzschild (linear effects in spin)

- Equivalence with the ADM result

  Extended the "contact" transformation

$$\mathbf{Y}_1 = \bar{\mathbf{x}}_1 + \frac{1}{c^3}\underset{S}{\mathbf{Y}}{}_1^{1.5\mathrm{PN}} + \frac{1}{c^4}\mathbf{Y}_1^{2\mathrm{PN}} + \frac{1}{c^5}\underset{S}{\mathbf{Y}}{}_1^{2.5\mathrm{PN}} + \frac{1}{c^6}\mathbf{Y}_1^{3\mathrm{PN}} + \frac{1}{c^7}\underset{S}{\mathbf{Y}}{}_1^{3.5\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

Reduction of the result

We first rewrite our result in term of spin variables of conserved Euclidian norm

Construct a variable $S^i[S^{\mu\nu}]$ such that $\delta_{ij} S^i S^j = s^2$
We propose a "canonical" construction but there is some rotation freedom!
The spin evolution equation now precession equation $\frac{\mathrm{d}\mathbf{S}_1}{\mathrm{d}t} = \mathbf{\Omega}_1 \times \mathbf{S}_1$ $\quad \mathbf{\Omega}_1 = \frac{1}{c^2}\mathbf{\Omega}_1^{\mathrm{1PN}} + \frac{1}{c^4}\mathbf{\Omega}_1^{\mathrm{2PN}} + \frac{1}{c^6}\mathbf{\Omega}_1^{\mathrm{3PN}} + \mathcal{O}\left(\frac{1}{c^7}\right)$

We need such conserved spins because they are secularly constant (for Taylor approximants)

We then reduce to the center of mass frame defined by $P^i = 0, G^i = 0$

$\longrightarrow$ everything is expressed in terms of n, v, r, and the spins

Finally, we are mostly interested in quasi-circular orbits

The emission of GW circularizes the orbit (Peters & Mathews formula)

We can look for solutions for which the separation r only varies du to radiation reaction $\dot{r} = \mathcal{O}(1/c^5)$

The expressions simplify a lot since $\mathbf{v} \cdot \mathbf{n} = \mathcal{O}(1/c^5)$

ω becomes a function of r

$\longrightarrow$ everything is expressed in terms of the spins and $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$

Reduced the result for the Energy

$$\underset{S}{E} = -\frac{m\nu c^2 x}{2}\left(\frac{x^{3/2}}{G\,m^2}\right)\left\{\frac{14}{3}S_\ell + 2\frac{\delta m}{m}\Sigma_\ell \right.$$

$$+x\left[\left(11-\frac{61}{9}\nu\right)S_\ell + \left(3-\frac{10}{3}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]$$

$$\left. +x^2\left[\left(\frac{135}{4}-\frac{367}{4}\nu+\frac{29}{12}\nu^2\right)S_\ell + \left(\frac{27}{4}-39\nu+\frac{5}{4}\nu^2\right)\frac{\delta m}{m}\Sigma_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right)\right\}$$

3.5 PN spin-orbit contribution to the binding energy

List of results for 3.5PN SO dynamics

We computed the spin-orbit contributions at NNLO order in terms of constant magnitude spins for general orbits of the

- Newtonian-like equations of motion of the bodies
- Spin precession equations
- conserved energy, linear and angular momenta, center of mass position
- Metric (in the near zone + regularized at the position of the bodies)
- Contact transformation that relates ADM and harmonic coord. orbital variables

We then reduced all these results to the center of mass frame, and specialized them to the case of quasi-circular orbits

see Class.Quant.Grav. 30 (2013)

Progress

- Motivation and introduction to PN
- Effective pole-dipole formalism: spinning point particles
- Sketch of the computation of the 3.5 PN spin-orbit effects
  - Equations of motion (and associated dynamical quantities)
  - Flux
- Estimates of the phase

Flux calculation

The flux can be expressed in terms of the (derivatives of) multipole moments

$$\mathcal{F} = \frac{G}{c^5}\left\{\frac{1}{5}I_{ij}^{(3)}I_{ij}^{(3)} + \frac{1}{c^2}\left[\frac{1}{189}I_{ijk}^{(4)}I_{ijk}^{(4)} + \frac{16}{45}J_{ij}^{(3)}J_{ij}^{(3)}\right] + \frac{1}{c^4}\left[\frac{1}{9072}I_{ijkl}^{(5)}I_{ijkl}^{(5)} + \frac{1}{84}J_{ijk}^{(4)}J_{ijk}^{(4)}\right] + \frac{1}{c^6}\left[\frac{4}{14175}J_{ijkl}^{(5)}J_{ijkl}^{(5)}\right] + (\text{tails}) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}$$

which can be expressed as integrals over
the matter and the gravitational fields in the source

$$J_L(t) = \mathop{\mathrm{FP}}_{B=0}\, \varepsilon_{ab<i_\ell} \int \mathrm{d}^3\mathbf{x}\, (r/r_0)^B \int_{-1}^{1} \mathrm{d}z \Big\{ \delta_\ell\, \hat{x}_{L-1>a}\, \Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\, \delta_{\ell+1}\, \hat{x}_{L-1>ac}\, \Sigma_{bc}^{(1)} \Big\}(\mathbf{x}, t + z\,r/c)\,,$$

$$\Sigma \equiv \frac{\tau^{00}+\tau^{ii}}{c^2}\,, \qquad \Sigma_i \equiv \frac{\tau^{0i}}{c}\,, \qquad \Sigma_{ij} \equiv \tau^{ij}\,,$$

Here again, the difficulty is to push 1PN order further all the spin-orbit contributions
in the sources of the integrals... (note that we need the equations of motion!)
... and to compute the new (regularized) integrals that appear!

Reduced the result for the flux

$$\underset{S}{\mathcal{F}} = \frac{32c^5}{5G}\, x^5\, \nu^2 \left(\frac{x^{3/2}}{G\, m^2}\right) \Bigg\{ -4S_\ell - \frac{5}{4}\frac{\delta m}{m}\Sigma_\ell$$
$$+x\left[\left(-\frac{9}{2}+\frac{272}{9}\nu\right)S_\ell + \left(-\frac{13}{16}+\frac{43}{4}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]$$
$$+x^{3/2}\left[-16\pi\, S_\ell - \frac{31\pi}{6}\,\frac{\delta m}{m}\Sigma_\ell\right]$$
$$+\,x^2\left[\left(\frac{476645}{6804}+\frac{6172}{189}\nu-\frac{2810}{27}\nu^2\right)S_\ell + \left(\frac{9535}{336}+\frac{1849}{126}\nu-\frac{1501}{36}\nu^2\right)\frac{\delta m}{m}\Sigma_\ell\right]$$
$$+\mathcal{O}\left(\frac{1}{c^5}\right)\Bigg\}\,. \qquad (3.13)$$

3.5 PN spin-orbit contribution to the emitted flux

Progress

- Motivation and introduction to PN
- Effective pole-dipole formalism: spinning point particles
- Sketch of the computation of the 3.5 PN spin-orbit effects
  - Equations of motion (and associated dynamical quantities)
  - Flux
- Estimates of the phase

Phase estimates

We can now apply the balance equation $$\frac{dE}{dt} = -\mathcal{F}$$

This becomes an evolution equation for x provided that the spins (the components orthogonal to the orbital plane) are constant over the radiation reaction timescale. It is the case with our constant magnitude spins at linear order in the spins.

Then $$\frac{dx}{dt} = -\frac{\mathcal{F}}{dE/dx}$$

We can re-expand the rhs (Taylor T2) and rewrite the integral defining the phase $\phi \equiv \int \omega \mathrm{d}t$ as an integral over x.

$$\begin{aligned}
\underset{S}{\phi} = -\frac{x^{-5/2}}{32\nu}\left(\frac{x^{3/2}}{G\,m^2}\right)\Bigg\{ & \frac{235}{6}S_\ell + \frac{125}{8}\frac{\delta m}{m}\Sigma_\ell \\
& + x\ln x\left[\left(-\frac{554345}{2016} - \frac{55}{8}\nu\right)S_\ell + \left(-\frac{41745}{448} + \frac{15}{8}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right] \\
& + x^{3/2}\left[\frac{940\pi}{3}S_\ell + \frac{745\pi}{6}\frac{\delta m}{m}\Sigma_\ell\right] \\
& + x^2\left[\left(-\frac{8980424995}{6096384} + \frac{6586595}{6048}\nu - \frac{305}{288}\nu^2\right)S_\ell\right. \\
& \qquad \left. + \left(-\frac{170978035}{387072} + \frac{2876425}{5376}\nu + \frac{4735}{1152}\nu^2\right)\frac{\delta m}{m}\Sigma_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right)\Bigg\}\,.
\end{aligned}$$

Phase estimates

TABLE I. Spin-orbit contributions to the number of gravitational-wave cycles $\mathcal{N}_{\rm GW} = (\phi_{\rm max} - \phi_{\rm min})/\pi$ accumulated from $\omega_{\rm min} = \pi \times 10\,{\rm Hz}$ to $\omega_{\rm max} = \omega_{\rm ISCO} = c^3/(6^{3/2} G m)$ for binaries detectable by ground-based detectors LIGO and VIRGO. For each compact object we define the magnitude χ_a and the orientation κ_a of the spin by $\mathbf{S}_a \equiv G\, m_a^2\, \chi_a\, \hat{\mathbf{S}}_a$ and $\kappa_a \equiv \hat{\mathbf{S}}_a \cdot \boldsymbol{\ell}$. For comparison, we give all the non-spin contributions up to 3.5PN order; however we neglect all the spin-spin terms.

	$1.4M_\odot + 1.4M_\odot$	$10M_\odot + 1.4M_\odot$	$10M_\odot + 10M_\odot$
Newtonian	15952.6	3558.9	598.8
1PN	439.5	212.4	59.1
1.5PN	$-210.3 + 65.6\kappa_1\chi_1 + 65.6\kappa_2\chi_2$	$-180.9 + 114.0\kappa_1\chi_1 + 11.7\kappa_2\chi_2$	$-51.2 + 16.0\kappa_1\chi_1 + 16.0\kappa_2\chi_2$
2PN	9.9	9.8	4.0
2.5PN	$-11.7 + 9.3\kappa_1\chi_1 + 9.3\kappa_2\chi_2$	$-20.0 + 33.8\kappa_1\chi_1 + 2.9\kappa_2\chi_2$	$-7.1 + 5.7\kappa_1\chi_1 + 5.7\kappa_2\chi_2$
3PN	$2.6 - 3.2\kappa_1\chi_1 - 3.2\kappa_2\chi_2$	$2.3 - 13.2\kappa_1\chi_1 - 1.3\kappa_2\chi_2$	$2.2 - 2.6\kappa_1\chi_1 - 2.6\kappa_2\chi_2$
3.5PN	$-0.9 + 1.9\kappa_1\chi_1 + 1.9\kappa_2\chi_2$	$-1.8 + 11.1\kappa_1\chi_1 + 0.8\kappa_2\chi_2$	$-0.8 + 1.7\kappa_1\chi_1 + 1.7\kappa_2\chi_2$

Of course, this is a very crude estimate of the importance of these new contributions in terms of actual searches and parameter estimation...

Conclusions

We have computed the NNLO SO effects (3.5PN for maximally spinning bodies) in the dynamics of the binary and in the emitted flux .

From a data analysis perspective, the main results are the new contributions to the binding energy and to the emitted flux since these increase the accuracy in Taylor approximants for the phase.

Crude estimates indicate that these terms should be relevant.

In addition to this, many other dynamical results which can be useful for comparisons with NR, building phenomenological models...

We are currently finishing the computation of the NLO tail term which seems to be of comparable magnitude.